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SUSTAINED p-DIVISIBLE GROUPS: FOLIATION OF MODULI SPACES REVISITED

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Introduction

Sustained p-divisible groups

Stabilized Hom schemes for p-divisible groups

Deformations of sustained p-divisible groups

Rigidity questions

Outline

Leaves in moduli spaces: origin

The notion of (central) leaves, due to Frans Oort, was announced in April 1999 (Texel).

The Hecke orbit conjecture, which states that every Hecke orbit is dense in the central leaf containing it, has stimulated the development of several methods and tools in arithmetic geometry.

Limitation of gfc

Originally central leaves are defined using the notion of geometrically fiberwise constant (gfc) p-divisible groups, a "pointwise" notion.

Will explain the definition of sustained p-divisible groups, a scheme-theoretic notion which updates gfc and reveals the fine structure of leaves. (joint with F. Oort)

Fix a prime number p from now on.

Review of p-divisible groups

SUSTAINED p-DIVISIBLE GROUPS: FOLIATION OF MODULI SPACES REVISITED

Ching-Li Chai

Introduction

Sustained p-divisible groups

Stabilized Hom schemes for p-divisible groups

Deformations of sustained p-divisible groups

Rigidity questions

Definition. A p-divisible group $X \to S$ over a base scheme S is a family $X = \big(X_n \to S\big)_{n\in\mathbb{N}}$ of finite locally free group schemes X_n killed by $[p^n]$, plus closed embeddings

$$\iota_{n+1,n} : X_n \to X_{n+1}$$

and faithfully flat homomorphisms

$$q_{n,n+1} : X_{n+1} \to X_n$$

such that

$$q_{n,n+1} \circ \iota_{n+1,n} = [p]_{X_n} \quad \forall n.$$

Idea: $X = \varinjlim_n X_n$ is divisible, with $X_n = \ker([p^n]_X) := X[p^n]$.

Hight of a p-divisible group

height: a p-divisible group $X \to S$ has a height

$$h : S \to \mathbb{N}$$

s.t.

$$\mathrm{rk}(X[p^n]) = p^{hn} \quad \forall n.$$

Slopes of a p-divisible group

SUSTAINED p-DIVISIBLE GROUPS: FOLIATION OF MODULI SPACES REVISITED

Ching-Li Chai

Introduction

Sustained p-divisible groups

Stabilized Hom schemes for p-divisible groups

Deformations of sustained p-divisible groups

Rigidity questions

slopes: a p-divisible group X of height h over a field $K \supseteq \mathbb{F}_p$ has h slopes in $\mathbb{Q} \cap [0,1]$, with multiplicities, s.t. $\mathrm{mult}(s) \cdot s \in \mathbb{N}$ for each slope s.

$d = \dim(X) =$ sum of slopes (with multiplicity)

Idea: a p-divisible group X is isoclinic of slope $\frac{a}{b}$ if the (bN)-th iterated Frobenius

$$\mathrm{Fr}_X^{bN} : X \to X^{(p^{bN})} \text{ is "approximately" } f_N \circ [p^{aN}]_X,$$

where $f_N : X \to X^{(p^{bN})}$ is "approximately an isomorphism".

Examples of p-divisible groups

1. If $A \to S$ is an abelian scheme, then $A[p^\infty] := \varinjlim_n A[p^n]$ is a p-divisible group over S.

Note. $A[p^\infty]$ is a version of p-adic $\mathrm{H}_1(A)$.

2. A p-divisible group $X \to S$ is isoclinic of slope 0 (respectively isoclinic of slope 1) iff $X[p^n] \to S$ is etale (respectively of multiplicative type) $\forall n$.

3. A p-divisible group is *ordinary* if its slopes $\subseteq \{0,1\}$. Elliptic curves with non-zero Hasse invariant are examples.

4. Isoclinic p-divisible groups with slope $1/2$ are said to be *supersingular*.

Polarization

SUSTAINED p-DIVISIBLE GROUPS: FOLIATION OF MODULI SPACES REVISITED

Ching-Li Chai

Introduction

Sustained p-divisible groups

Stabilized Hom schemes for p-divisible groups

Deformations of sustained p-divisible groups

Rigidity questions

Definition. (i) A polarization of an abelian scheme $A \to S$ is a homomorphism $\lambda : A \to A^t$ such that $\lambda^t = \lambda$ and $\lambda_{\bar{s}}$ comes from an ample invertible $\mathscr{O}_{A_{\bar{s}}}$-module, for every geometric point $\bar{s}$ of S. Here A^t is the abelian scheme dual to A

(ii) A polarization of a p-divisible group $X \to S$ is a homomorphism $\lambda : X \to X^t$ such that $\lambda^t = \lambda$ and $\mathrm{Ker}(\lambda)$ is finite locally free over S.

Geometrically fiberwise constant p-divisible groups

Definition. A p-divisible group X over a scheme S in char. p is geometrically fiberwise constant if any two fibers X_{s_1}, X_{s_2} are isomorphic when based-changed to a common algebraically closed field K which contains both $\kappa(s_1)$ and $\kappa(s_2)$.

Similarly for a *polarized* p-divisible group $(X \to S, \lambda : X \to X^t)$.

Note. p-divisible groups are not rigid: they don't have coarse moduli spaces—otherwise gfc is a stupid (and useless) notion.

Central leaves via gfc

Original defn: a central leaf in the moduli space $\mathscr{A}_g$ of g-dimensional principally polarized abelian varieties in char. p is a maximal element among the family of all reduced locally closed subvariety Z of $\mathscr{A}_g$ s.t. the universal p-divisible group over Z is geometrically fiberwise constant.

Each central leaf is a smooth locally closed subvariety of $\mathscr{A}_g$, stable under all prime-to-p Hecke correspondences.

But this definition through gfc, which is a "point-wise" notion, is handicapped. (E.g. what's the deformation functor for leaves?)

Example of central leaves

1. The ordinary locus $\mathscr{A}_g^{\mathrm{ord}}$ in $\mathscr{A}_g$, which is a dense open subset, is a leaf.

2. Every supersingular leaf in $\mathscr{A}_g$ is finite.

3. Every central leaves in $\mathscr{A}_3$ with slopes $\{1/3, 2/3\}$ is 2-dimensional; it has a natural structure as a torsor for an isoclinic p-divisible group of height 6 and slope $1/3$.

Strongly κ-sustained p-divisible groups

SUSTAINED p-DIVISIBLE GROUPS: FOLIATION OF MODULI SPACES REVISITED
Ching-Li Chai
Introduction
Sustained p-divisible groups
Stabilized Hom schemes for p-divisible groups
Deformations of sustained p-divisible groups
Rigidity questions

$\kappa \supset \mathbb{F}_p$: a field of char. $p > 0$
S/κ: a κ-scheme
Y/κ: a p-divisible group

Definition. A p-divisible group $X \to S$ is strongly κ-sustained modeled on Y if $\forall n > 0$, the Isom scheme

$$\mathbf{Isom}_S(Y[p^n] \times_{\mathrm{Spec}(\kappa)} S, X[p^n]) \longrightarrow S$$

is faithfully flat.[1]

Similarly one has the notion of a strongly κ-sustained polarized p-divisible group $(X \to S, \mu_X : X \to X^t)$ modeled on a polarized p-divisible group (Y, μ_Y) over κ.

[1] musical origin of the terminology: sostenuto

Existence of slope filtration

Proposition 1. Let $X \to S/\kappa$ be a strongly κ-sustained p-divisible group. There exists a unique slope filtration

$$(0) = \mathrm{Fil}_0 X \subsetneqq \mathrm{Fil}_1 X \subsetneqq \cdots \subsetneqq \mathrm{Fil}_m X = X$$

of X by p-divisible subgroups such that

(1) $\mathrm{Fil}_i X/\mathrm{Fil}_{i-1} X$ is a κ-sustained p-divisible group, isoclinic of slope s_i for $i = 1, \ldots, m$.

(2) $1 \geq s_1 > s_2 > \cdots > s_m \geq 0$.

Backward compatibility with gfc

SUSTAINED p-DIVISIBLE GROUPS: FOLIATION OF MODULI SPACES REVISITED

Ching-Li Chai

Introduction

Sustained p-divisible groups

Stabilized Hom schemes for p-divisible groups

Deformations of sustained p-divisible groups

Rigidity questions

Theorem 2. Suppose that S is a reduced scheme over a field $\kappa \supseteq \mathbb{F}_p$. Let $X \to S/\kappa$ and Y/κ be p-divisible groups. If X_s is strongly κ-sustained modeled on Y $\forall s \in S$, then $X \to S$ is strongly κ-sustained modeled on Y/κ.

Updated definition of leaves

Definition. A (central) leaf in $\mathscr{A}_g$ over $\overline{\mathbb{F}_p}$ is a maximal element among the family of all locally closed subschemes $Z \subseteq \mathscr{A}_g$ over $\overline{\mathbb{F}_p}$ with the following property:

The principally polarized p-divisible group $\big((\boldsymbol{A},\boldsymbol{\mu})|_Z\big)[p^\infty]$ attached to the restriction to Z of the universal principally polarized abelian scheme $(\boldsymbol{A},\boldsymbol{\mu})$ is strongly $\overline{\mathbb{F}_p}$-sustained.

Stabilized Hom schemes for p-divisible groups

SUSTAINED p-DIVISIBLE GROUPS: FOLIATION OF MODULI SPACES REVISITED

Ching-Li Chai

Introduction

Sustained p-divisible groups

Stabilized Hom schemes for p-divisible groups

Deformations of sustained p-divisible groups

Rigidity questions

Given Y,Z: p-divisible groups over a field $\kappa \supset \mathbb{F}_p$.
Define group schemes of finite type over κ

$$H_n := \mathbf{Hom}(Y[p^n], Z[p^n])$$

We have arrows

- $r_{n,n+i}\colon H_{n+i} \to H_n$ (restriction homomorphism)
- $\iota_{n+i,n}\colon H_n \hookrightarrow H_{n+i}$ (induced by $[p^i]_{H_{n+i}}$)

Define

$$\mathbf{Hom}^{\mathrm{st}}(Y,Z)_n := \mathrm{Im}(r_{n,n+i}\colon H_{n+i} \to H_n) \text{ for } i \gg 0$$

We have

$$\mathbf{Hom}^{\mathrm{st}}(Y,Z)_{n+1} \underset{j_{n+1,n}}{\overset{\pi_{n,n+1}}{\rightleftarrows}} \mathbf{Hom}^{\mathrm{st}}(Y,Z)_n$$

Stabilized Hom schemes, continued

Theorem 3.
(a) $\mathbf{Hom}^{\mathrm{st}}(Y,Z) := \big(\mathbf{Hom}^{\mathrm{st}}(Y,Z)_{n+1}, j_{n+1,n}, \pi_{n,n+1}\big)_{n\in\mathbb{N}}$
is a p-divisible group over κ.

(b) If the field $\kappa \supseteq \mathbb{F}_p$ is perfect, then the Dieudonné module

$$\mathbb{D}_*\big(\mathbf{Hom}^{\mathrm{st}}(Y,Z)\big)$$

is the largest $W(\kappa)$-submodule of $\mathrm{Hom}_{W(\kappa)}\big(\mathbb{D}_*(Y),\mathbb{D}_*(Z)\big)$ which is stable under the semi-linear operators F and V.

(c) Suppose that Y,Z are isoclinic over κ, with slopes s_Y and s_Z respectively.

- If $s_Y > s_Z$, then $\mathbf{Hom}^{\mathrm{st}}(Y,Z) = (0)$.
- If $s_Y \le s_Z$, then $\mathbf{Hom}^{\mathrm{st}}(Y,Z)$ is isoclinic of slope $s_Z - s_Y$ and height $\mathrm{ht}(Z)\cdot\mathrm{ht}(Y)$.

The projective system of stabilized Aut groups

Definition. Given a p-divisible group Y over $\kappa \supseteq \mathbb{F}_p$, we have

- $\big(\mathbf{End}^{\mathrm{st}}(Y)_n, \pi_{n,n+1} : \mathbf{End}^{\mathrm{st}}(Y)_{n+1} \to \mathbf{End}^{\mathrm{st}}(Y)_n\big)_{n\in\mathbb{N}}$, a projective system of finite ring schemes over κ, and
- $\Gamma(Y)_\bullet := \big(\mathbf{End}^{\mathrm{st}}(Y)_n^\times, \pi_{n,n+1}\big)_{n\in\mathbb{N}}$, a projective system of finite group schemes over κ.

Observation.

$$\big(\text{strongly } \kappa\text{-sustained } p\text{-divisible groups modeled on } Y\big)$$
$$\longleftrightarrow \big(\text{projective systems of right } \Gamma(Y)_\bullet\text{-torsors}\big)$$

$$(X \to S) \quad \rightsquigarrow \quad \big(\mathbf{Isom}^{\mathrm{st}}(Y[p^n]\times S, X[p^n])\big)_n$$

SUSTAINED p-DIVISIBLE GROUPS: FOLIATION OF MODULI SPACES REVISITED

Ching-Li Chai

Introduction
Sustained p-divisible groups
Stabilized Hom schemes for p-divisible groups
Deformations of sustained p-divisible groups
Rigidity questions

Smoothness of sustained deformations

Let $\mathbf{Def}^{\mathrm{sus}}(Y)$ be the deformation functor which to every Artinian local ring $(R,\mathfrak{m})$ over κ assigns the set of isomorphism classes of strongly sustained p-divisible groups $\mathfrak{X}$ over R whose closed fiber is Y.

Theorem 4. The functor $\mathbf{Def}^{\mathrm{sus}}(Y)$ is smooth over κ.
Proof. Set-up:

- Let $(R,\mathfrak{m})$ and $(R',\mathfrak{m}')$ be Artinian local κ-algebras, $\kappa = R'/\mathfrak{m}'$, $R = R'/J$, $J\cdot\mathfrak{m}' = (0)$; i.e. R' is a small extension of $S := \mathrm{Spec}(R)$. Let $S_0 := \mathrm{Spec}(\kappa)$
- $\left(T_n\right)_{n\in\mathbb{N}}$: a projective system of right $\Gamma(Y)_\bullet$-torsors over S.

Want to show: The T_n's extends to a projective system of $\Gamma(Y)_\bullet$-torsors over $S' = \mathrm{Spec}(R')$.

Proof of Theorem 4 continued

Sustained p-divisible groups: foliation of moduli spaces revisited

Ching-Li Chai

Introduction

Sustained p-divisible groups

Stabilized Hom schemes for p-divisible groups

Deformations of sustained p-divisible groups

Rigidity questions

Illusie:

(1) We have a perfect complex $\ell_{T_n/S}$ of $\mathscr{O}_S$-modules of amplitude $\subseteq [-1,0]$, the co-Lie complex of T_n/S.

(2a) The obstruction of lifting T_n to S' is an element of $H^2(S,\ell^\vee_{T_n/S}\otimes^{\mathbb{L}}_R J)) \cong H^2(S_0,\ell^\vee_{T_n\times_S S_0/S_0}\otimes_\kappa J) = (0)$.

(2b) The set of isomorphism classes of all liftings of T_n to S' is a torsor for
$H^1(S_0,\ell^\vee_{T_n\times_S S_0/S_0}\otimes_\kappa J) =: \nu_{T_n\times_S S_0/S_0}\otimes_\kappa J$.

The slope filtration on $\mathbf{End}^{\mathrm{st}}(Y)$ gives a filtration on $\Gamma(Y)_\bullet$ whose successive quotients, except the first/etale part, come from p-divisible groups. Devissage gives:

$$\nu_{T_{n+1}\times_S S_0/S_0}\otimes_\kappa J \xrightarrow{\sim} \nu_{T_n\times_S S_0/S_0}\otimes_\kappa J.$$

(Used: $\nu_{Z[p^{n+1}]/\kappa} \xrightarrow{\sim} \nu_{Z[p^n]/\kappa}\ \forall\, p$-divisible group Z over κ.) $\square$

Application to central leaves

Remarks. (a) A similar argument shows that the deformation functor $\mathbf{Def}^{\mathrm{sus}}(Y, \lambda)$ of a polarized p-divisible group with arbitrary polarization degree is smooth.

(b) It follows that every central leaf in the modular variety $\mathscr{A}_{g,d}$ is smooth, for any polarization degree d. Similarly for all PEL modular varieties.

Local linear structure of leaves

Tate-linear structure.
Example 1. Case when z_0 corresponds to an abelian variety with two slopes $s_1 > s_2$.
Then $\mathscr{C}^{/z_0}$ has a natural structures as a torsor for an isoclinic p-divisible group of slope $s_1 - s_2$.

Example 2. Case when z_0 corresponds to an abelian variety whose p-divisible group is a product of three isoclinic p-divisible groups with slopes $s_1 > s_2 > s_3$.
Then $\mathscr{C}^{/z_0}$ has a natural structures as a biextension of p-divisible groups $Z_{12} \times Z_{23}$ by a p-divisible group Z_{13}, isoclinic of slopes $s_1 - s_2, s_2 - s_3$ and $s_1 - s_3$ respectively.

Local linear structure of leaves, continued

Phenomenon. The formal completion at a closed point of a leaf $\mathscr{C} \subseteq \mathscr{A}_g$ has a natural Tate-linear structure in the following sense:

it is “built up” from p-divisible groups, through a finite family of fibrations, each of which is a torsor for a p-divisible group.

Local rigidity: preliminary definitions

Sustained p-divisible groups: foliation of moduli spaces revisited

Ching-Li Chai

Introduction

Sustained p-divisible groups

Stabilized Hom schemes for p-divisible groups

Deformations of sustained p-divisible groups

Rigidity questions

Definition. Let $\kappa \supseteq \mathbb{F}_p$ be a field.
(a) An action of a p-adic Lie group G on a p-divisible formal group Y over κ is strongly non-trivial if none of the Jordan–Hölder component the induced action of the Lie algebra $\mathrm{Lie}(G)$ of G on $\mathbb{D}_*(Y)_{\mathbb{Q}}$ is the trivial representation of $\mathrm{Lie}(G)$.

(b) An action of a p-adic Lie group G on a biextension of p-divisible formal groups $Y_1 \times Y_2$ by a p-divisible formal group Z over κ is strongly non-trivial if the induced actions of G on Y_1, Y_2 and Z are all strongly non-trivial.

Local rigidity: examples

Theorem 5. Let $\kappa \supseteq \mathbb{F}_p$ be a field and let G be a p-adic Lie group. Let Y_1, Y_2, Z be p-divisible formal groups over κ.

(1) If $V \subseteq Z$ is an irreducible closed formal subvariety of Z stable under a strongly non-trivial action of G on Z, then V is a formal subgroups of Z.

(2) Let B be a bi-extension of $Y_1 \times Y_2$ by Z Suppose that $V \subset B$ is an irreducible closed formal subvariety of B stable under a strongly non-trivial action of G on B. Then V is a Tate-linear subvariety of B. Furthermore if Y_1, Y_2 do not have any slope in common, then V is a sub-biextension of B.

Rigidity questions

SUSTAINED p-DIVISIBLE GROUPS: FOLIATION OF MODULI SPACES REVISITED

Ching-Li Chai

Introduction

Sustained p-divisible groups

Stabilized Hom schemes for p-divisible groups

Deformations of sustained p-divisible groups

Rigidity questions

Let $\mathscr{C} \subseteq \mathscr{A}_g$ be a leaf in $\mathscr{A}_g$ over $\overline{\mathbb{F}_p}$. Let $Z \subset \mathscr{C}$ be an irreducible closed subscheme of $\mathscr{C}$ over $\overline{\mathbb{F}_p}$, and let $z \in Z(\overline{\mathbb{F}_p})$ be a closed point of Z.

Suppose that $Z^{/z} \subseteq$ is stable under a strongly non-trivial action of a p-adic Lie group G which respects the Tate-linear structure on $\mathscr{C}^{/z}$.

Expectation 1. (local rigidity) Z is Tate-linear at $z \in \mathscr{C}$, i.e. $Z^{/z} \subset \mathscr{C}^{/z}$ is Tate-linear.

Expectation 2. Z is Tate-linear at every closed point of Z.

Question 3. Is Z the reduction of a Shimura subvariety of $\mathscr{A}_g$?