Classification of p-divisible groups over rings of mixed characteristic

Timo Henkel

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1. Remarks on p-divisible Groups
2. A Classification Theorem for p-divisible Groups over $\mathcal{O}_C$
3. Applications

p-divisible groups

Convention

1. Throughout this talk we fix a complete and algebraically closed extension C of $\mathbb{Q}_p$. We denote the integral closure of $\mathbb{Z}_p$ in C by $\mathcal{O}_C$ (take for example $C = \mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$).
2. G denotes a p-divisible group over $\mathcal{O}_C$, i.e. an inductive system of group schemes $(G[p^1] \to G[p^2] \to \dots)$ such that $G[p^n] = \mathrm{Spec}(A_n)$ (where the A_n are finite and locally free $\mathcal{O}_C$-algebras) satisfying several conditions.

Example

$\mu_{p^\infty} = (\mu_p \to \mu_{p^2} \to \dots)$, where $\mu_{p^n} = \mathrm{Spec}(\mathcal{O}_C[X]/(X^{p^n} - 1))$ and hence $\mu_{p^n}(S) = \{x \in S \mid x^{p^n} = 1\}$ for any $\mathcal{O}_C$-algebra S.

Adification of p-divisible groups and its generic fibre

Remark

1. To a p-divisible group G one can associate an adic space G^{ad}.
2. G^{ad} is an adic space over $\mathcal{O}_C$, i.e. there exists a canonical morphism of adic spaces $G^{ad} \to \mathrm{Spa}(\mathcal{O}_C, \mathcal{O}_C)$.
3. Define the *generic fibre* of G^{ad} to be the fibre product $G^{ad}_\eta := G^{ad} \times_{\mathrm{Spa}(\mathcal{O}_C,\mathcal{O}_C)} \mathrm{Spa}(C, \mathcal{O}_C)$.
4. G^{ad}_η is an analytic adic space.
5. This happens in a similar way in which a variety over $\mathbb{C}$ can be turned into a $\mathbb{C}$-analytic space.
6. Whereas a finite scheme is described by polynomial equations, analytic spaces are given by restrictions of formal power series.

Cartier-duality

Proposition

1. For a finite group scheme $H = \text{Spec}(A)$ over $\mathcal{O}_C$ we find a finite group scheme $H^* = \text{Spec}(A^*)$ over $\mathcal{O}_C$ with $A^* = \text{Hom}_{\mathcal{O}_C}(A, \mathcal{O}_C)$. H^* is called the *Cartier-dual* of H.
2. For a p-divisible group $G = (\text{Spec}(A_1) \to \text{Spec}(A_2) \to \dots)$ over $\mathcal{O}_C$ there is a *Cartier-dual* $G^* = (\text{Spec}(A_1^*) \to \text{Spec}(A_2^*) \to \dots)$.

Example

The Cartier-dual of $G = \mu_{p^\infty}$ is $G^* = \mathbb{Q}_p/\mathbb{Z}_p$ and we have $(\mathbb{Q}_p/\mathbb{Z}_p)^* = \mu_{p^\infty}$.

The Lie algebra of a p-divisible group

Proposition

There exists a functor from the category of p-divisible groups over $\mathcal{O}_C$ to the category of locally free $\mathcal{O}_C$-modules; $G \mapsto \operatorname{Lie} G$.

Proposition

The *dimension* of G is the $\mathcal{O}_C$-rank of $\operatorname{Lie} G$. We have

$$\operatorname{height}(G) = \operatorname{height}(G^*) = \dim(G) + \dim(G^*).$$

Example

i) $\operatorname{Lie} \mathbb{Q}_p/\mathbb{Z}_p = \{0\}$.
ii) Since $\mu_{p^\infty} = (\mathbb{Q}_p/\mathbb{Z}_p)^*$, $\operatorname{rank}(\operatorname{Lie} \mu_{p^\infty}) = \dim(\mu_{p^\infty}) = 1$.
iii) The p-divisible group $G' := \mathbb{Z}_p^n \otimes_{\mathbb{Z}_p} \mu_{p^\infty} := (\mu_{p^\infty})^n$ has Lie algebra $\operatorname{Lie} G' = \mathcal{O}_C^n$.

Vector group and logarithm

Proposition (Scholze and Weinstein)

1. To Lie G we can associate an adic space Lie $G \otimes \mathbb{G}_a$ which has the structure of an $\mathbb{Z}_p$-module.
2. There exists a $\mathbb{Z}_p$-linear map $\log_G : G_\eta^{ad} \to \text{Lie } G \otimes \mathbb{G}_a$.
3. $\log_G$ is a local isomorphism.

Remark

1. Lie $G \otimes \mathbb{G}_a$ contains the same information as Lie $G \otimes_{\mathcal{O}_C} C$.
2. Since $p \in C$ is topologically nilpotent, $G_\eta^{ad} \xrightarrow{\cdot p} G_\eta^{ad}$ is topologically nilpotent. Hence, by p-adically completeness of C, we have a canonical $\mathbb{Z}_p$-module structure on G_η^{ad}.

Tate module reviewed

Proposition

Let G be a p-divisible group over $\mathcal{O}_C$. The *Tate module* of G

$$T(G) = \varprojlim_n G[p^n](C)$$

is a free $\mathbb{Z}_p$-module of rank height(G).

Example

i) $T(\mathbb{Q}_p/\mathbb{Z}_p) = \varprojlim_n \underline{\mathbb{Z}/p^n\mathbb{Z}}(C) = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}_p$.

ii) $T(\mu_{p^\infty}) = \varprojlim_n \mu_{p^n}(C) = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}_p$.

iii) For $G' = \mathbb{Z}_p^n \otimes_{\mathbb{Z}_p} \mu_{p^\infty}$ as before, we have $T(G') = \mathbb{Z}_p^n$.

Upcoming questions

Question

How much of the information of G can be recovered from $T(G)$?

Example

As $\mathbb{Z}_p$-modules, we have $T(\mu_{p^\infty}) = T(\mathbb{Q}_p/\mathbb{Z}_p)$ but μ_{p^∞} and $\mathbb{Q}_p/\mathbb{Z}_p$ are not isomorphic as p-divisible groups over $\mathcal{O}_C$:
$\dim(\mathbb{Q}_p/\mathbb{Z}_p) = 0 \neq 1 = \dim(\mu_{p^\infty})$

Question

Is G uniquely determined by the pair $(T(G), \mathrm{Lie}\, G)$?

Hodge-Tate exact sequence

Theorem (Fargues)

There is a natural short exact sequence of C-vector spaces:

$$0 \to \operatorname{Lie} G \otimes_{\mathcal{O}_C} C \xrightarrow{\alpha^*_{G^*}} T(G) \otimes_{\mathbb{Z}_p} C \xrightarrow{\alpha_G} (\operatorname{Lie} G^*)^* \otimes_{\mathcal{O}_C} C \to 0$$

Remark

1. Via the exact sequence above, we can consider $\operatorname{Lie} G \otimes_{\mathcal{O}_C} C$ as a linear subspace of $T(G) \otimes_{\mathbb{Z}_p} C$.
2. This way we can simultaneously use the data of the Tate module and the Lie algebra of a p-divisible group.

Equivalence of categories

Definition

Let $(\{T, W\})$ be the category of pairs consisting of a free $\mathbb{Z}_p$-module T of finite rank and a C-linear subspace $W \subseteq T \otimes_{\mathbb{Z}_p} C$.

The Hodge-Tate exact sequence gives a functor F

$$\begin{array}{ccc} (p\text{-divisible groups over } \mathcal{O}_C) & \longrightarrow & (\{T, W\}) \\ G & \mapsto & (T(G), \mathrm{Lie}\, G \otimes_{\mathcal{O}_C} C), \end{array}$$

where we consider $\mathrm{Lie}\, G \otimes_{\mathcal{O}_C} C$ as a subspace of $T \otimes_{\mathbb{Z}_p} C$ via $\alpha^*_{G^*}$.

Theorem (Scholze and Weinstein)

F is an equivalence of categories.

Proof of the theorem

Plan

Find a functor $J : \{(T, W)\} \to$ (p-divisible groups $/\mathcal{O}_C$) such that J and F are quasi-inverse to each other.

Step 1

1. We consider the p-divisible group $G' := T \otimes_{\mathbb{Z}_p} \mu_{p^\infty}$ for which we have $T(G') = T$ and $\operatorname{Lie} G' = T \otimes_{\mathbb{Z}_p} \mathcal{O}_C$.
2. G' is a p-divisible group with $\operatorname{height}(G') = \dim(G') = rank(T)$.
3. We have the natural inclusion $W \hookrightarrow T \otimes_{\mathbb{Z}_p} C = \operatorname{Lie} G' \otimes_{\mathcal{O}_C} C$.

Proof of the theorem

Step 2

We define H to be an adic space, making the following diagram Cartesian

$$\begin{array}{ccc} H & \longrightarrow & W \otimes \mathbb{G}_a \\ \downarrow & & \downarrow \\ G'^{ad}_{\eta} & \longrightarrow & \text{Lie}\, G' \otimes \mathbb{G}_a. \end{array}$$

The vertical maps can be considered as inclusions and the horizontal maps are logarithm maps.

Proof of the theorem

Step 3

We define G as the following disjoint union:

$$G = \coprod_{Y \subseteq H} \mathrm{Spf}(\mathrm{H}^0(Y, \mathcal{O}_Y^+)),$$

where Y ranges over all connected components of H.
(Remember that $H = G_\eta'^{ad} \times_{\mathrm{Lie}\, G' \otimes \mathbb{G}_a} W \otimes \mathbb{G}_a$.)

One can show that G is in fact a p-divisible group over $\mathcal{O}_C$ with $G_\eta^{ad} \cong H$, $\mathrm{Lie}\, G \otimes_{\mathcal{O}_C} C \cong W$ and $T(G) \cong T(G') = T$. Moreover it can be proven that be obtain a functor J which is quasi-inverse to F in this way.

Applications

Consequences

i) Up to isomorphism, there are exactly two p-divisible groups over $\mathcal{O}_C$ of height 1.

ii) $\mathbb{Q}_p/\mathbb{Z}_p$ corresponds to $(\mathbb{Z}_p, 0)$ and μ_{p^∞} to $(\mathbb{Z}_p, C)$.

iii) Let $d \leq h$ be non-negative integers. The problem to find all p-divisible groups of height h and dimension d is equivalent to the problem to find all d-dimensional linear subspaces of C^h.

iv) The Grassmannian variety $\mathrm{Gras}(d, C^h)$ gets involved.

Connection to Dieudonné Modules

Remark

The map $\mathcal{O}_C \to k := \mathcal{O}_C/p\mathcal{O}_C$ induces a *base change functor*

$$(p\text{-divisible groups over } \mathcal{O}_C) \longrightarrow (p\text{-divisible groups over } k)$$

Outlook

We obtain the following commutative diagram of categories:

$$\begin{array}{ccc} (p\text{-divisible groups over } \mathcal{O}_C) & \xrightarrow{\cong} & (\{T, W\}) \\ \downarrow & & \downarrow ??? \\ (p\text{-divisible groups over } k) & \xrightarrow{\cong} & (\text{Dieudonné modules}) \end{array}$$