Hodge-Tate decomposition for p-divisible groups

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April 2020

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1 Preliminaries

In this section, we introduce some objects and results that will be useful later on.

1.1 Formal Lie algebra

Let R be a ring and consider $\mathcal{A} = R[[X_1, \ldots, X_n]]$ the ring of power series in n variables over R.

Definition 1.1. *A commutative formal group law of dimension n over R is an n-tuple of power series $F = (F_1, \ldots, F_n)$, $F_i \in R[[X_1, \ldots, X_n, Y_1, \ldots Y_n]]$ such that*

1. $F_i(\underline{X}, 0) = F_i(0, \underline{X}) = X_i$,
2. $F_i(F(\underline{X}, \underline{Y}), Z) = F_i(Z, F(\underline{X}, \underline{Y}))$,
3. $F_i(\underline{X}, \underline{Y}) = F_i(\underline{Y}, \underline{X})$.

It turns out that $1)$ and $2)$ automatically imply the existence of an inverse map $\iota : R[[X_1, \ldots, X_n]] \to R[[X_1, \ldots, X_n]]$ given by power series $\iota_i(\underline{X})$ that verifies

$$F(\underline{X}, \iota(\underline{X})) = F(\iota(\underline{X}), \underline{X}) = 0.$$

A p-divisible formal Lie group is a commutative formal Lie group F such that the map

$$[p] : R[[X_1, \ldots, X_n]] \to R[[X_1, \ldots, X_n]],$$

makes $R[[X_1, \ldots, X_n]]$ a free module over itself.
Let $I = (X_1, \ldots, X_n)$ be the augmentation ideal of $\mathcal{A}$. If F is a p-divisible formal Lie group, for $v \geqslant 1$, consider the ring $A_v = \mathcal{A}/[p^v]_F(I)$. It is a finite flat R-module and F equips the scheme $\Gamma_v = \operatorname{Spec} A_v$ with the structure of a group scheme. There are also canonical inclusions $i_v : \Gamma_v \to \Gamma_{v+1}$ and one can verify that (Γ_v, i_v) forms a connected p-divisible group. We have the following theorem:

Theorem 1.2. *Let R be a complete local Noetherian ring with residue characteristic $p > 0$. Then the above construction $F \rightsquigarrow (\Gamma_v, i_v)$ is an equivalence between the category of p-divisible formal Lie groups and the category of connected p-divisible groups.*

Remark 1.3. *Note that if $G = (G_v = \operatorname{Spec} A_v, i_v)$ is a connected p-divisible group, the arrow in the other direction is given by equipping $\mathcal{A} \cong \varprojlim_v A_v$ with the formal group structure induced from the compatible group structures on the G_v.*

If $(G_v = \operatorname{Spec} A_v, i_v)$ is a p-divisible group over a complete local Noetherian ring R, there exists a compatible family of exact sequences

$$0 \to G_v^\circ \to G_v \to G_v^{\text{ét}} \to 0,$$

which gives rise to an exact sequence of p-divisible groups

$$0 \to G^\circ \to G \to G^{\text{ét}} \to 0,$$

where G° is a connected p-divisible group called the connected component of G, and $G^{\text{ét}}$ is an étale p-divisible group called the étale component of G. By the previous theorem, we have that $\mathcal{A}^\circ = \varprojlim_v A_v^\circ \cong R[[X_1, \cdots, X_n]]$, where $n = \dim G$. If S is an R-algebra, we define the tangent space of G with coefficients in S to be

$$\mathfrak{t}_G(S) = \operatorname{Hom}_R(I/I^2, S),$$

where I is the augmentation ideal of $\mathcal{A}^\circ$.

1.2 Points of a p-divisible group

Let $\mathcal{O}$ be a complete Noetherian local domain with maximal ideal $\mathfrak{m}$ and fraction field K, and consider $G = (G_v = \operatorname{Spec} A_v, i_v)_v$ a p-divisible group over $\mathcal{O}$. If R is a complete local $\mathcal{O}$-algebra, we want to define the set of R-valued points of G in a way that it carries topological information. In order for this to work, we need to assume that for all $x \in \mathfrak{m}_R$, there exists $r \gg 0$ such that $x^r \in \mathfrak{m}R$.
We define the R-valued points of G to be

$$G(R) = \varprojlim_r \varinjlim_v G_v(R/\mathfrak{m}^r R).$$

Putting $\mathcal{A} = \varprojlim_v A_v$ equipped with the inverse limit topology (where each A_v has the $\mathfrak{m}A_v$-adic topology), we see that

$$G(R) = \varprojlim_r \varinjlim_v \operatorname{Hom}_{\mathcal{O}\text{-alg}}(A_v, R/\mathfrak{m}^r R) = \operatorname{Hom}_{\text{Cont } \mathcal{O}\text{-alg}}(\mathcal{A}, R),$$

where R has the $\mathfrak{m}R$-adic topology.

Lemma 1.4. *The following properties are true:*

1. *$G(R)$ is a $\mathbb{Z}_p$-module.*
2. *The canonical map $\varinjlim_v \varprojlim_r G_v(R/\mathfrak{m}^r R) \to \varprojlim_r \varinjlim_v G_v(R/\mathfrak{m}^r R)$ is injective and its image is $G(R)_{tors}$ (the torsion points of $G(R)$).*
3. *If G is étale then the previous map is an isomorphism, $G(R) = \varinjlim_v G_v(R/\mathfrak{m}_R)$ and it is torsion.*
4. *If $\mathcal{O}$ has residue characteristic p and G is connected, then there exists a (non-canonical) G_K-equivariant isomorphism (as topological spaces)*
$$G(R) \cong \mathfrak{m}_R^n,$$
where n is the dimension of G.

Proposition 1.5. *Suppose that $\mathcal{O}$ has perfect residue field of characteristic p and let R be a complete local $\mathcal{O}$-algebra, then the sequence of Abelian groups*

$$0 \to G^\circ(R) \to G(R) \to G^{\acute{e}t}(R) \to 0$$

is exact.

Corollary 1.6. *If moreover R is normal and has algebraically closed residue field, then $G(R)$ is a divisible group.*

1.3 The Tate module

With the same notation as before, let K^{alg} be the algebraic closure of K and let G_K be the absolute Galois group $\text{Gal}(K^{\text{alg}}/K)$ of K. The Tate module of G is the $\mathbb{Z}_p[G_K]$-module

$$T_p(G) := \varprojlim_v G_v(K^{\text{alg}}),$$

where the limit is taken over the maps $j_v : G_{v+1} \to G_v$ induced by the multiplication by p. The Tate co-module is the $\mathbb{Z}_p[G_K]$-module

$$\Phi_p(G) := \varinjlim_v G_v(K^{\text{alg}}),$$

where the colimit is taken over the inclusions i_v.

Lemma 1.7. *The Tate module and co-module have the following properties:*

1. *$T_p(G)$ is a free $\mathbb{Z}_p$-module of rank h such that $T_p(G)/p^v = G_v(K^{alg})$,*
2. *$\Phi_p(G) \cong T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$ as $\mathbb{Z}_p[G_K]$ modules (so $\Phi_p(G) \cong (\mathbb{Q}_p/\mathbb{Z}_p)^h$ as $\mathbb{Z}_p$-modules) and $\Phi_p(G)[p^v] = G_v$,*
3. *$T_p(G) \cong \text{Hom}_{\mathbb{Z}_p}\big(\mathbb{Q}_p/\mathbb{Z}_p, \Phi_p(G)\big)$ as $\mathbb{Z}_p[G_K]$-modules.*

Proposition 1.8. *Let K be any field of characteristic 0. The functor $G \rightsquigarrow T_p(G)$ induces an equivalence of categories*

$$\left\{ \begin{array}{c} p\text{-divisible groups} \\ \text{over } K \end{array} \right\} \xrightarrow{\cong} \left\{ \begin{array}{l} \text{finite free } \mathbb{Z}_p\text{-modules with} \\ \text{continuous } \mathbb{Z}_p\text{-linear action of } G_K \end{array} \right\}$$

2 Hodge-Tate decomposition for p-divisible groups

In this section, we consider a complete discrete valuation field K with valuation ν. We let $\mathcal{O}$ be its ring of integers, whose maximal ideal and residue field are denoted respectively by $\mathfrak{m}$ and k. The valuation ν on K extends uniquely to K^{alg}, as well as $\mathbb{C}_K$, the completion of K^{alg} under ν. We denote by G_K the absolute Galois group of K and by R the ring of integers of $\mathbb{C}_K$. We will work with a p-divisible group $G = (G_v, i_v)$ over $\mathcal{O}$.

2.1 The logarithm map

Much like in the case of Lie groups, we can define a logarithm map, relating points on our p-divisible group with its tangent space, which is functorial in G and satisfies the expected nice properties. Concretely, this map is $\mathbb{Z}_p$-module homomorphism given by

$$\log_G : G(R) \to \mathfrak{t}_G(\mathbb{C}_K) = \text{Hom}_{\mathcal{O}}(I/I^2, \mathbb{C}_K),$$
$$f \mapsto \big(x \mapsto \lim_{r\to\infty} \frac{([p^r]f)(x)}{p^r}\big).$$

First note that since $G^{\text{ét}}(R)$ is torsion, and since we have an exact sequence

$$0 \to G^\circ(R) \to G(R) \to G^{\text{ét}}(R) \to 0,$$

then $[p^r]f \in G^\circ(R) = \mathrm{ContHom}_{\mathcal{O}-\mathrm{alg}}(\mathcal{A}^\circ, V)$ for $r \gg 0$. Hence $([p^r]f)(x)$ makes sense for big r.
We define a filtration on $G^\circ(R)$ given by

$$F^\lambda G^\circ(R) := \{f \in G^\circ(R) \mid \nu\big(f(a)\big) \geqslant \lambda \forall a \in I\}.$$

If $f \in F^\lambda G^\circ(R)$ and $x \in I$, then $[p]x = px + z$ for some $z \in I^2$, and

$$([p]f)(x) = f([p]x) = f(px + z) = pf(x) + f(z).$$

Since f is multiplicative, we have $\nu(f(z)) \geqslant 2\lambda$. Hence $\nu([p]f(x)) \geqslant \min(\nu(p) + \lambda, 2\lambda)$ and we get

$$[p]F^\lambda G^\circ(R) \subseteq F^{\lambda+\min(\nu(p),\lambda)} G^\circ(R).$$

Since any $f \in G^\circ(R)$ belongs to $F^\lambda G^\circ(R)$ for some $\lambda > 0$, up to replacing f by $[p^i]f$ for some $i \gg 0$, we can assume that $\lambda \geqslant \nu(p)$. Replacing f with $[p^r]f$ in the previous calculations, we get

$$\frac{([p^{r+1}]f)(x)}{p^{r+1}} - \frac{([p^r]f)(x)}{p^r} = \frac{[p^r](f)(z)}{p^{r+1}},$$

which has valuation $\geqslant 2(\lambda + r\nu(p)) - (r+1)\nu(p) = 2^\lambda + (r-1)\nu(p) \to \infty$ as $r \to \infty$.
This shows that the sequence $\frac{([p^r]f)(x)}{p^r}$ is Cauchy, so it converges in $\mathbb{C}_K$. It also shows that if $z \in I^2$, then the sequence $\frac{([p^r]f)(z)}{p^r}$ tends to zero as $r \to \infty$. Therefore $\log_G$ is well defined.
The logarithm $\log_G$ is actually a group homomorphism. Indeed, if we let $f, g \in G^\circ(R)$, we have

$$f + g = \mathcal{A} \xrightarrow{F} \mathcal{A}\widehat{\otimes}\mathcal{A} \xrightarrow{f\otimes g} R\widehat{\otimes}R \xrightarrow{\mu} R,$$

where for $x \in I$, $F(x) = x \otimes 1 + 1 \otimes x + z$ with $z \in I\widehat{\otimes}I$. Hence $(f+g)(x) \equiv f(x) + g(x) \mod f(I)g(I)$. And we get that $\log_G(f+g) = \log_G(f) + \log_G(g)$ using the above calculations.

Lemma 2.1. *We have the following properties:*

1. *The logarithm $\log_G : G(R) \to \mathfrak{t}_G(\mathbb{C}_K)$ is a local homeomorphism. More precisely, for every $\lambda > \frac{\nu(p)}{p-1}$ it induces an isomorphism*
$$\log_G : F^\lambda G^\circ(R) \xrightarrow{\cong} \{\tau \in \mathfrak{t}_G(\mathbb{C}_K) \mid \nu(\tau(X_i)) \geqslant \lambda \textit{ for } 1 \leqslant i \leqslant n\}.$$

2. *We have a short exact sequence*

$$0 \to G(R)_{tors} \to G(R) \xrightarrow{\log_G} \mathfrak{t}_G(\mathbb{C}_K) \to 0.$$

Proof. **1)** Let $\tau \in \mathfrak{t}_G(\mathbb{C}_K)$ be an element of the set on the right hand side. We want to construct $f \in F^\lambda G^\circ(R)$ such that $\log_G(f) = \tau$.
Let $\mu = \nu(p)$. We can write $[p](\underline{X}) = p\big(\underline{X} + \phi(\underline{X})\big) + \psi(\underline{X})$ with $\deg(\psi) \geqslant p$ and $\deg(\phi) \geqslant 2$ (see the Corollary to Lazard's theorem in [5, p. 115]). An easy calculation shows that for $\underline{x} \in \mathfrak{m}_{\mathbb{C}_K}^n$ $\nu(\underline{x}) > \frac{\mu}{p-1}$, $\nu\big([p](\underline{x})\big) = \nu(\underline{x}) + \nu$.
In fact multiplication by p induces an isomorphism $[p] : F^\lambda G^\circ(R) \xrightarrow{\cong} F^{\lambda+\mu} G^\circ(R)$ for all $\lambda > \frac{\mu}{p-1}$(see Theorem 4 in [5, p. 119]). Therefore, for every $r \geqslant 0$ there is a unique element $f_r \in F^\lambda G^\circ(R)$ such that $([p^r]f_r)(X_i) = p^r\tau(X_i)$ for all $1 \leqslant i \leqslant n$. And we have

$$\begin{aligned} [p^{r+1}]\big(f_r(X_i) - f_{r+1}(X_i)\big) &= [p]\big(p^r\tau(X_i)\big) - p^{r+1}\tau(X_i) \\ &= p\phi(p^r\tau(X_i)) + \psi(p^r\tau(X_i)). \end{aligned}$$

Thus $\nu\big([p^{r+1}]\big(f_r(X_i) - f_{r+1}(X_i)\big)\big) \geqslant (2r+1)\mu + 2\lambda$, and $\nu\big(f_r(X_i) - f_{r+1}(X_i)\big) \geqslant r\mu + 2\lambda \to \infty$ as $r \to \infty$.
Finally, we set $f(X_i) = \lim\limits_{r\to\infty} f_r(X_i)$ for $1 \leqslant i \leqslant n$. Almost by construction, we have

$$\lim_{r\to\infty} \frac{([p^r]f)(X_i)}{p^r} = \tau(X_i).$$

Thus $\log_G(f) = \tau$. And this constitute the inverse of the logarithm.
2) Since $\mathfrak{t}_G(\mathbb{C}_K)$ is torsion free, we have that $G(R)_{\text{tors}} \subseteq \ker \log_G$. Conversely, if $f \in \ker \log_G$, then $[p^i]f \in \ker \log_G$ for all $i \geqslant 0$. But for $i \gg 1$, we have $[p^i]f \in F^\lambda G^\circ(R)$, so by the statement above, we must have $[p^i]f = 0$ and f is torsion. To prove surjectivity of $\log_G$, let $\tau \in \mathfrak{t}_G(\mathbb{C}_K)$, then for $i \gg 1$, $p^i\tau$ lies in the right-hand-side of the above isomorphism, so $p^i\tau$ lies in the image of $\log_G$. But by corollary 1.6, $G(R)$ is divisible, so τ is also in the image of $\log_G$ which proves the result. □

Example 2.2. *If $G = \mu_{p^\infty,\mathcal{O}}$ is the p-divisible group associated to $\mathbb{G}_m$, one can verify that $G(R) = 1 + \mathfrak{m}_R$, so $G(R)_{tors}$ is the group μ_{p^∞} of p-power roots of unity in $1 + \mathfrak{m}_R$, and we have an exact sequence*

$$0 \to \mu_{p^\infty} \to 1 + \mathfrak{m}_R \xrightarrow{\log_G} \mathbb{C}_K \to 0,$$

where $\log_G(1 + x) = \sum_{n=1}^{\infty}(-1)^{n+1}x^n/n$ is the usual logarithm.

2.2 Exploiting duality

Using the Cartier duality, we can define a p-divisible group $G^D = (G_v^D, j_v^D)_v$ where G_v^D is the Cartier dual of G_v and $j_v : G_{v+1} \to G_v$ is the map induced by the multiplication by p . We have pairings $G_v^D \times G_v \to \mu_{p^v}$ for each $v \geqslant 1$ which are compatible in the sense that the following diagram

$$\begin{array}{ccccc} G_{t+v}^D & \times & G_{t+v} & \to & \mu_{p^{t+v}} \\ \Big\downarrow{\scriptstyle j_{t,v}} & & {\scriptstyle i_{v,t}}\Big\uparrow & & {\scriptstyle i_v}\Big\uparrow \\ G_v^D & \times & G_v & \longrightarrow & \mu_{p^v} \end{array}$$

commutes. Taking the K^{alg}-points and then the limit over v projectively with G_v^D and inductively with G_v yields a G_K-equivariant perfect pairing

$$T_p(G^D) \times \Phi_p(G) \to \Phi_p(\mu_{p^\infty}) = \mathbb{Q}_p/\mathbb{Z}_p(1). \tag{1}$$

We also get another G_K-equivariant perfect pairing

$$T_p(G^D) \times T_p(G) \to T_p(\mu_{p^\infty}) = \mathbb{Z}_p(1), \tag{2}$$

where we add (1) for the twist by the cyclotomic character. Indeed, for $(x_v)_v \in T_p(G^D)$ and $(y_v)_v \in T_p(G)$, we have

$$\langle x_v, y_v\rangle = \langle j_{t,v}(x_{t+v}), j_{t,v}(y_{t+v})\rangle = \langle x_{t+v}, i_{v,t}j_{t,v}(y_{t+v})\rangle = \langle x_{t+v}, p^t(y_{t+v})\rangle = \langle x_{t+v}, y_{t+v}\rangle^{p^t}.$$

So we set

$$\langle (x_v)_v, (y_v)_v\rangle = (\langle x_v, y_v\rangle)_v \in T_p(\mu_{p^\infty}),$$

which is a level-wise G_K-equivariant perfect pairing.

2.3 The Hodge-Tate decomposition

Given that every finite flat group scheme over an algebraically closed field of characteristic 0 is constant, we have for every $v \geqslant 1$

$$G_v(K^{\text{alg}}) = G_v(\mathbb{C}_K) = G_v(R).$$

The last equality holds since every $\mathcal{O}$-algebra homomorphism $A \to \mathbb{C}_K$ with A finite as an $\mathcal{O}$-module factors through R. Thus, Cartier duality yields an isomorphism

$$T_p(G^D) = \varprojlim_v G_v^D(R) = \varprojlim_v \mathrm{Hom}_{\mathrm{Gsch}/R}(G_v \otimes_{\mathcal{O}} R, \mu_{p^v,R}) = \mathrm{Hom}_{p\text{-divisible groups}}(G \otimes_{\mathcal{O}} R, \mu_{p^\infty,R}),$$

of $\mathbb{Z}_p[G_K]$-modules. By functoriality, any map of p-divisible groups induces a map on the points and on the tangent spaces over $\mathbb{C}_K$. Hence, we get two G_K-equiavariant maps

$$T_p(G^D) \to \mathrm{Hom}_{\mathbb{Z}_p}(G(R), \mu_{p^\infty}(R)) \quad \text{and} \quad T_p(G^D) \to \mathrm{Hom}_{\mathbb{C}_K}\big(\mathfrak{t}_G(\mathbb{C}_K), \mu_{p^\infty,\mathcal{O}}(\mathbb{C}_K)\big).$$

These induce $\mathbb{Z}_p[G_K]$-module homomorphisms

$$\alpha : G(R) \to \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^D), 1+\mathfrak{m}_R),$$

which restricts to torsion points, and

$$d\alpha : \mathfrak{t}_G(\mathbb{C}_K) \to \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^D), \mathbb{C}_K).$$

Proposition 2.3. *We have a map of exact sequences of $\mathbb{Z}_p[G_K]$-modules*

$$\begin{tikzcd}
0 \arrow[r] & G(R)_{tors} \arrow[r] \arrow[d, "\alpha_0"] & G(R) \arrow[r, "\log_G"] \arrow[d, "\alpha"] & \mathfrak{t}_G(\mathbb{C}_K) \arrow[r] \arrow[d, "d\alpha"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^D), \mu_{p^\infty}) \arrow[r] & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^D), 1+\mathfrak{m}_R) \arrow[r, "\log_*"] & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^D), \mathbb{C}_K) \arrow[r] & 0
\end{tikzcd}$$

where α_0 is an isomorphism, and α and $d\alpha$ are injective.

Proof. First, note that the square commutes by functoriality of the logarithm and by the definition of α and $d\alpha$. Moreover, the bottom map is $\mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^D), -)$ applied to the exact sequence

$$0 \to \mu_{p^\infty} \to 1+\mathfrak{m}_R \xrightarrow{\log} \mathbb{C}_K \to 0.$$

So it's exact ($T_p(G^D)$ is a free $\mathbb{Z}_p$-module). Next, using the isomorphism $G(R)_{\text{tors}} \cong \Phi_p(G)$, α_0 identifies with the isomorphism $\Phi_p(G) \cong \mathrm{Hom}_{\mathbb{Z}_p}\left(T_p(G^D), \Phi_p(\mu_{p^\infty})\right)$ induced by the perfect pairing (1). Thus α_0 is an isomorphism. It remains to show that α and $d\alpha$ are injective. We will prove this in a series of steps:

Step 1 The kernel and cokernel of α are $\mathbb{Q}_p$-vector spaces (A priori they are only $\mathbb{Z}_p$-modules).
Applying the snake lemma and the fact that α_0 is an isomorphism, we get that the maps $\ker\alpha \to \ker d\alpha$ and $\operatorname{coker}\alpha \to \operatorname{coker} d\alpha$ induced by the logarithm are isomorphisms. But since $d\alpha$ is a $\mathbb{Z}_p$-linear map between $\mathbb{Q}_p$-vector spaces it is automatically $\mathbb{Q}_p$-linear. Thus its kernel and cokernel are also $\mathbb{Q}_p$-vector spaces.

Step 2 We have $G(R)^{G_K} = G(\mathcal{O})$ and $\mathfrak{t}_G(\mathbb{C}_K)^{G_K} = \mathfrak{t}_G(K)$.
By the exact sequence in proposition 1.5, it suffices to show the fact separately for $G^{\text{ét}}$ and G°. By $4.$ of lemma 3.2 and Ax-Sen-Tate theorem, we have

$$G^\circ(R)^{G_K} = (\mathfrak{m}_R^d)^{G_K} = \mathfrak{m}^d = G^\circ(\mathcal{O}).$$

Again by $3.$ of lemma 3.2, we have $G^{\text{ét}}(R) = \varinjlim_v G_v(k^{\text{alg}})$ and $G^{\text{ét}}(\mathcal{O}) = \varinjlim_v G_v(k)$ where $k^{\text{alg}} = R/\mathfrak{m}_R$ is the algebraic closure of k. But on any group scheme H over $\mathcal{O}$, we have $H((k^{\text{alg}})^{G_K}) = H(k^{\text{alg}})^{G_K} = H(k)$ from which we get the result.
For the tangent spaces, by Ax-Sen-Tate theorem,

$$\mathfrak{t}_G(\mathbb{C}_K)^{G_K} = \mathrm{Hom}_{\mathcal{O}}(I/I^2, \mathbb{C}_K^{G_K}) = \mathrm{Hom}_{\mathcal{O}}(I/I^2, K) = \mathfrak{t}_G(K).$$

Step 3 α is injective on $G(\mathcal{O})$.
Since $G(R)^{G_K} = G(\mathcal{O})$, the kernel of α restricted to $G(\mathcal{O})$ is $(\ker\alpha)^{G_K}$ which is a $\mathbb{Q}_p$ vector space by step 1. Decomposing G into its étale and connected pieces, by a diagram chase (and the fact that $T_p((G^\circ)^D) \twoheadrightarrow T_p(G^D)$) we can reduce to these pieces. So it suffices to shows that neither $G^{\text{ét}}(\mathcal{O})$ nor $G^\circ(\mathcal{O})$ contain a non-zero $\mathbb{Q}_p$-vector space. But this is straightforward for $G^{\text{ét}}(\mathcal{O})$ since it is torsion (by lemma 3.2). For $G^\circ(\mathcal{O})$, since the valuation on $\mathcal{O}$ is discrete, we have $[p]F^\delta G^\circ(\mathcal{O}) \subseteq F^{\delta+1}G^\circ(\mathcal{O})$. Moreover, $\bigcap_v [p^v]G^\circ(\mathcal{O}) = 0$, so $G^\circ(\mathcal{O})$ does not contain a non-zero p^∞-divisible point. Thus it cannot contain a non-zero $\mathbb{Q}_p$-vector space.

Step 4 The restriction of $d\alpha$ to $\mathfrak{t}_G(K) \subseteq \mathfrak{t}_G(\mathbb{C}_K)$ is injective.
Using a diagram chase and the fact that α is injective on $G(\mathcal{O})$ we get that $d\alpha$ is injective on $\log_G(G(\mathcal{O}))$. Thus it is also injective on the $\mathbb{Q}_p$-vector space it generates. But this is exactly $\mathfrak{t}_G(K)$. Indeed for any $\tau \in \mathfrak{t}_G(K)$, there exist $n \gg 0$ such that $p^n\tau \in \operatorname{im}\log_G(G(\mathcal{O}))$ by lemma 2.1.

Step 5 The map $d\alpha$ is injective (hence so is α).
We factor it as follows

$$\mathfrak{t}_G(\mathbb{C}_K) = \mathfrak{t}_G(K) \otimes_K \mathbb{C}_K \to \operatorname{Hom}_{\mathbb{Z}_p}(T_p(G^D), \mathbb{C}_K)^{G_K} \otimes_K \mathbb{C}_K \to \operatorname{Hom}_{\mathbb{Z}_p}(T_p(G^D), \mathbb{C}_K),$$

where the first map is injective by step 4 and the second map is injective by the following step:

Step 6 Let W be a $\mathbb{C}_K$-vector space endowed with a semi-linear $\mathbb{C}_K$-action. Then, the $\mathbb{C}_K$-linear map

$$f : W^{G_K} \otimes_K \mathbb{C}_K \to W,$$

is injective.
Suppose that this is not the case, and let $w \neq 0 \in \ker f$, say $w = w_1 \otimes c_1 + w_2 \otimes c_2 \cdots + w_r \otimes c_r$ with $w_i \in W^{G_K}$ and $c_i \in \mathbb{C}_K$. We can assume that r is minimal (among the expressions of elements of the kernel of f), so in particular $c_i \neq 0$. Up to dividing x by c_1, we can also assume that $c_1 = 1$. If $\sigma \in G_K$, $\sigma(w) = w_1 \otimes 1 + w_2 \otimes \sigma(c_2) + \cdots + w_r \otimes \sigma(c_r) \in \ker f$. And $\sigma(w) - w \in \ker f$ is a tensor of rank $\leqslant r-1$ so it must be zero by minimality of r. If $\sigma(c_i) \neq c_i$ for some i, then $w_i \otimes 1$ is a linear combination of the other $w_j \otimes 1$ which contradicts the minimality of r. So by Ax-Sen-Tate theorem, $c_i \in \mathbb{C}_K^{G_K} = K$. Thus $r = 1$, and we get $f(w_1 \otimes 1) = 0 \implies w_1 = 0$. □

Theorem 2.4. *The maps*

$$\alpha_{\mathcal{O}} : G(\mathcal{O}) \to \operatorname{Hom}_{\mathbb{Z}_p[G_K]}(T_p(G^D), 1 + \mathfrak{m}_R),$$

and

$$d\alpha_{\mathcal{O}} : \mathfrak{t}_G(K) \to \operatorname{Hom}_{\mathbb{Z}_p[G_K]}(T_p(G^D), \mathbb{C}_K),$$

induced from the previous maps by taking the G_K-invariants, are isomorphisms.

Proof. Let us recollect what we know from the previous proposition in the following diagram

$$\begin{tikzcd}
0 \arrow[r] & G(R)_{\text{tors}} \arrow[r] \arrow[d, "\cong"', "\alpha_0"] & G(R) \arrow[r, "\log_G"] \arrow[d, hook, "\alpha"] & \mathfrak{t}_G(\mathbb{C}_K) \arrow[r] \arrow[d, hook, "d\alpha"] & 0 \\
0 \arrow[r] & \operatorname{Hom}_{\mathbb{Z}_p}(T_p(G^D), \mu_{p^\infty}) \arrow[r] & \operatorname{Hom}_{\mathbb{Z}_p}(T_p(G^D), 1 + \mathfrak{m}_R) \arrow[r, "\log_*"] \arrow[d, two heads] & \operatorname{Hom}_{\mathbb{Z}_p}(T_p(G^D), \mathbb{C}_K) \arrow[r] \arrow[d, two heads] & 0 \\
 & & \operatorname{coker}\alpha \arrow[r, "\cong"] & \operatorname{coker} d\alpha &
\end{tikzcd}$$

Taking the G_K-invariants of the two vertical columns, we get two exact sequences

$$0 \to G(\mathcal{O}) \xrightarrow{\alpha_{\mathcal{O}}} \operatorname{Hom}_{\mathbb{Z}_p[G_K]}\left(T_p(G^D), 1 + \mathfrak{m}_R\right) \to (\operatorname{coker}\alpha)^{G_K},$$

and

$$0 \to \mathfrak{t}_G(K) \xrightarrow{d\alpha_{\mathcal{O}}} \operatorname{Hom}_{\mathbb{Z}_p[G_K]}\left(T_p(G^D), \mathbb{C}_K\right) \to (\operatorname{coker} d\alpha)^{G_K}.$$

It follows that the map $\operatorname{coker}\alpha_{\mathcal{O}} \to \operatorname{coker} d\alpha_{\mathcal{O}}$ induced by the logarithm, is injective. Therefore, it is enough to show that $d\alpha_{\mathcal{O}}$ is surjective.
Since $d\alpha_{\mathcal{O}}$ is an injective morphism of K-vector spaces, we just need to equate the dimension of both sides. For this, we set

$$W = \operatorname{Hom}_{\mathbb{Z}_p}(T_p(G), \mathbb{C}_K) \quad \text{and} \quad W^D = \operatorname{Hom}_{\mathbb{Z}_p}(T_p(G^D), \mathbb{C}_K).$$

Let $d = \dim_K W^{G_K}$ and $d' = \dim_K (W^D)^{G_K}$. By injectivity of $d\alpha_{\mathcal{O}}$, we have $\dim G = \dim_K \mathfrak{t}_G(K) \leqslant d'$, and swapping G and G^D, we also have $\dim G^D \leqslant d$. We want to show that the equalities hold, but since $\dim G + \dim G^D = \mathrm{ht}(G) =: h$, it suffices to show that $d + d' \leqslant h$. The key is to use the Tate modules pairing 2 which induces a perfect pairing of h-dimensional $\mathbb{C}_K$-vector spaces

$$W \times W^D \to \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p(1), \mathbb{C}_K) \cong \mathbb{C}_K(-1). \tag{3}$$

Taking the G_K-invariants, we get a pairing

$$W^{G_K} \times (W^D)^{G_K} \to \mathrm{Hom}_{\mathbb{Z}_p[G_K]}(\mathbb{Z}_p(1), \mathbb{C}_K) = 0,$$

where the vanishing is due to Ax-Sen-Tate's theorem ($\mathbb{C}_K$ does not contain a period for the cyclotomic character). We deduce that $W^{G_K} \otimes_K \mathbb{C}_K$ and $(W^D)^{G_K} \otimes_K \mathbb{C}_K$ are orthogonal with respect to the pairing (3) so the sum of their dimensions is $\leqslant h$ as desired. ☐

Corollary 2.5. *(Hodge-Tate decomposition)*
There is a G_K-equivariant isomorphism of $\mathbb{C}_K$-vector spaces

$$T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{C}_K \cong \big(\mathfrak{t}^*_{G^D}(K) \otimes_K \mathbb{C}_K\big) \oplus \big(\mathfrak{t}_G(K) \otimes_K \mathbb{C}_K(1)\big),$$

where $\mathfrak{t}^*_{G^D}(K)$ is the cotangent space of G^D at k, i.e., $\mathfrak{t}^*_{G^D}(K) = \mathrm{Hom}_K(\mathfrak{t}_{G^D}(K), K)$.

Proof. Actually the proof of the previous theorem, tells us that $d + d' = h$, so $W^{G_K} \otimes_K \mathbb{C}_K$ and $(W^D)^{G_K} \otimes_K \mathbb{C}_K$ are complements of each other under the pairing (3). Therefore, by the isomorphism in theorem 2.4, we get a G_K-equivariant exact sequence

$$0 \to \mathfrak{t}_G(K) \otimes_K \mathbb{C}_K \xrightarrow{d\alpha_{\mathcal{O}} \otimes \mathrm{id}} W^D \to \mathrm{Hom}_{\mathbb{Z}_p}\big(\mathfrak{t}_{G^D}(K) \otimes_K \mathbb{C}_K, \mathbb{C}_K(-1)\big) \to 0,$$

tensoring by $\mathbb{C}_K(1)$, and noting that $T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{C}_K = \mathrm{Hom}_{\mathbb{Z}_p}\big(T_p(G^D), \mathbb{Z}_p(1)\big) \otimes_{\mathbb{Z}_p} \mathbb{C}_K = W^D(1)$, we get

$$0 \to \mathfrak{t}_G(K) \otimes_K \mathbb{C}_K(1) \to T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{C}_K \to \mathfrak{t}^*_{G^D}(K) \otimes_K \mathbb{C}_K \to 0.$$

But by Ax-Sen-Tate's theorem, we have that $\mathrm{Ext}^1_{G_K}\big(\mathbb{C}_K, \mathbb{C}_K(1)\big) = H^1\big(G_K, \mathbb{C}_K(1)\big) = 0$ so the above sequence splits which gives the result. ☐

3 The main theorem

In this section, we will prove the following result

Theorem 3.1. *Let R be an integrally closed, Noetherian integral domain, whose fraction field K is of characteristic 0. Let G and H be p-divisible groups over R, then the map*

$$\mathrm{Hom}_{R\text{-pdiv}}(G, H) \to \mathrm{Hom}_{K\text{-pdiv}}(G \otimes_R K, H \otimes_R K),$$

is bijective.

Before attempting to prove this theorem, we shall make a few reductions. We begin by reformulating the problem, so for $v \geqslant 1$, consider A_v (resp. B_v) to be the Hopf algebra associated to G_v (resp. H_v). An element $\varphi \in \mathrm{Hom}_{K\text{-pdiv}}(G \otimes_R K, H \otimes_R K)$ is a compatible sequence of morphisms of group schemes $\varphi_v : G_v \otimes_R K \to H_v \otimes_R K$, or equivalently a compatible sequence of morphisms of Hopf algebras $u_v : B_v \otimes_R K \to A_v \otimes_R K$. Note that since B_v is flat over R, the injectivity of the map in the statement of the theorem is immediate, so the hard part is to prove surjectivity.
Since B_v and A_v are finitely generated as an R-module, we can identify u_v with a matrix with coefficients in K relative to a generating system over R of B_v and A_v. To ask that u_v lifts to a morphism $B_v \to A_v$ is the same as asking that the matrix has coefficients in R. So if we know that the result holds for discrete valuation rings, then it holds for any localisation $R_{\mathfrak{p}}$ of R with respect to a minimal prime ideal $\mathfrak{p}$. Consequently, the matrix associated to u_v

has coefficients in $R_{\mathfrak{p}}$ for all minimal prime ideals $\mathfrak{p}$. But since R is integrally closed, we have $\bigcap_{\mathfrak{p} \text{ minimal}} R_{\mathfrak{p}} = R$, so u_v has coefficients in R as desired. Therefore, we can reduce to the case that R is a discrete valuation ring. Note that if the residue characteristic is zero, then the p-divisible groups are étale and the theorem follows (we have an explicit description of étale p-divisible groups over Henselian local rings). Therefore, for the rest of the section, we can assume that R is a complete discrete valuation ring with residue field of characteristic $p > 0$.

Lemma 3.2. *If $g : G \to H$ is a homomorphism of p-divisible groups over R such that its restriction $G \otimes_R K \to H \otimes_R K$ is an isomorphism, then g is an isomorphism.*

Proof. Let $G = (G_v)_v$, $H = (H_v)_v$ and A_v (resp. B_v) be the Hopf algebra corresponding to G_v (resp. H_v). We have a sequence of compatible homomorphisms $u_v : B_v \to A_v$ such that $u_v \otimes \mathrm{id} : B_v \otimes_R K \to A_v \otimes_R K$ are isomorphism. Given that B_v is flat over R, it follows that u_v is injective for all $v \geqslant 1$. Now B_v and A_v are finite flat modules over R (a noetherian local ring), so they are free of rank $k = p^{vh}$ where h is the height of G (and H). Moreover B_v is a submodule of A_v, and since R is a PID, there exist a basis $\omega_1, \dots, \omega_k$ of A_v over R such that $\pi^{r_1}\omega_1, \dots, \pi^{r_2}\omega_k$ is a basis of B_v (where π is the uniformizer of R). If we can show that $\mathrm{disc}(B_v) = \mathrm{disc}(A_v)$ then we are done. Indeed, we have

$$\det\big(\mathrm{Tr}(\pi^{r_i}\omega_i \pi^{r_j}\omega_j)\big) = c^2 \det\big(\mathrm{Tr}(\omega_i\omega_j)\big),$$

where $c = \pi^{\sum_i r_i}$. So in that case we would have $c \in R^\times$ which implies $r_i = 0$ for all i, and consequently that $A_v = B_v$.

We have a formula for the discriminant of a p-divisible group given by $\mathrm{disc}(G_v) = \big(p^{nvp^{hv}}\big)$ where n is the dimension of G and h is its height. Since G and H have the same height (it is determined by the Tate module, hence by the generic fiber), in order for the equality between the discriminants to hold, it suffices to show that they have the same dimension. But the dimension is also determined by the Tate module thanks to Theorem 2.4 ($\dim G = \dim_K \mathfrak{t}_G(K)$). So G_v and H_v have the same discriminant, which proves the lemma. ∎

Proposition 3.3. *If F is a p-divisible group over R, and M is a G_K-submodule of $T_p(F)$ which is a $\mathbb{Z}_p$-direct summand, then there exist a p-divisible group E over R and a homomorphism $\varphi : E \to F$ inducing closed immersions at each finite level and an isomorphism $T_p(E) \cong M$ via $T_p(\varphi)$.*

Proof. By proposition 1.8, the module $M \subseteq T_p(F)$ corresponds to a p-divisible group E^* over K which is a closed subgroup of $F \otimes_R K$. For $v \geqslant 1$, let $F_v = \mathrm{Spec}(B_v)$, $E_v^* = \mathrm{Spec}\, A_v^*$, so that we have a surjective morphism of rings $u_v : B_v \otimes_R K \to A_v^*$. We set $A_v = u_v(B_v)$ and $E'_v = \mathrm{Spec}\, A_v$. Then E'_v is a finite flat closed R-subgroup scheme of F_v, and $E_v^* = E'_v \otimes_R K$.

The inclusions $F_v \hookrightarrow F_{v+1}$ induce inclusions $u'_v : E'_v \hookrightarrow E'_{v+1}$. Since orders can be computed on the generic fiber, each E'_v has order p^{vh} where h is the $\mathbb{Z}_p$-rank of M. Moreover, given that E^* is a p-divisible group, E_v^* is killed by p^v, hence so is E'_v. Similarly, we get that E'_{v+1}/E'_v is killed by p. Therefore, multiplication by p induce homomorphisms

$$E'_{v+2}/E'_{v+1} \to E'_{v+1}/E'_v, \tag{4}$$

which become isomorphisms on the generic fiber. Set $E'_{v+1}/E'_v = \mathrm{Spec}\, D_v$, then by definition of surjectivity of maps between finite flat group schemes, the morphism $D_v \to A_{v+1}$ is faithfully flat, so in particular it is injective. Therefore D_v is finitely generated as an R-module. The morphism in (4) corresponds to a morphism $D_v \to D_{v+1}$ which becomes an isomorphism upon tensoring with K, so it is injective, and as a consequence, the D_v can be viewed as an increasing sequence of R-orders in the finite étale K-algebra $D_1 \otimes_R K$. Given that the D_v are finite as R-modules, they all lie in the integral closure of R inside $D_1 \otimes_R K$ which we denote $\widetilde{R}$. But R being an integrally closed Noetherian domain, $\widetilde{R}$ is a Noetherian R-module. Therefore, the increasing sequence of R-modules $(D_v)_v$ inside $\widetilde{R}$ is stationary, i.e., there exist $v_0 \geqslant 1$, such that $\forall v \geqslant v_0$, $D_v = D_{v+1}$.

Define $E_v = E'_{v+v_0}/E'_{v_0}$. The inclusions $u'_{v+v_0} : E'_{v+v_0} \hookrightarrow E'_{v+v_0+1}$ induce inclusions $u_v : E_v \hookrightarrow E_{v+1}$, and we now show that (E_v, u_v) is a p-divisible group. In order to do so, let us consider the following diagram

$$\begin{tikzcd}
E_{v+1} = E'_{v+v_0+1}/E'_{v_0} \arrow[r, "p^v"] \arrow[d, "\alpha"] & E'_{v+v_0+1}/E'_{v_0} = E_{v+1} \\
E'_{v_0+v+1}/E'_{v_0+v} \arrow[r, "\beta"] & E'_{v_0+1}/E'_{v_0} \arrow[u, "\gamma"']
\end{tikzcd}$$

where α is the canonical projection, γ the canonical inclusion, and β is the composition of the multiplication by p maps as in (4). By the choice of v_0, β is an isomorphism, and given that γ is a closed immersion, we have $\ker[p^v] = \ker \alpha = E'_{v_0+v}/E'_{v_0} = E_v$. Moreover, the order of E_v is equal to $\mathrm{ord}(E'_{v_0+v})/\mathrm{ord}(E'_{v_0}) = p^{vh}$. So (E_v, u_v) is a p-divisible group of order h as desired.
Finally, we have a morphism of p-divisible groups $\varphi : E \to F$ given on each finite level by

$$E_v = E'_{v_0+v}/E'_{v_0} \xrightarrow{p^{v_0}} E'_v \subseteq F_v,$$

which identifies $T_p(E)$ with $T_p(E^*) = M$. $\square$

Finally, we finish by the proof of the main theorem.

Proof. of Theorem 3.1.
Let G and H be p-divisible groups over R, and let $f : G \otimes_R K \to H \otimes_R K$ be a morphism of p-divisible groups between their generic fibers. Consider the graph Γ of $T_p(f)$ inside $T_p(G) \times T_p(H)$ which is a G_K-module. The quotient $\big(T_p(G) \times T_p(H)\big)/\Gamma$ injects into $T_p(H)$ via $(x, y) \mapsto y - T_p(f)(x)$, so it is torsion free, hence a free $\mathbb{Z}_p$-module. Consequently, Γ is a direct $\mathbb{Z}_p$ summand of $T_p(G) \times T_p(H)$. Applying proposition 3.3 to Γ, we get a p-divisible group $E \subseteq G \times H$ over R such that $T_p(E) \cong \Gamma$. Therefore, the projection $\pi_1 : E \to G$ induces an isomorphism $T_p(E) \cong T_p(G)$, hence an isomorphism on the generic fibers (by proposition 1.8). So by lemma 3.2, π_1 is an isomorphism. Finally, the morphisms of p-divisible groups over R

$$\pi_2 \circ \pi_1^{-1} : G \to H,$$

extends f (one can see that by looking for example at the map on the Tate modules). This proves the surjectivity of our map, hence the theorem. $\square$

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