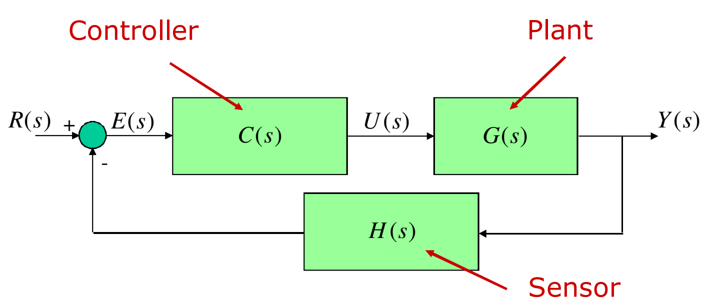
ME 343 – Control Systems

Lecture 15

September 25, 2009

Root Locus



$$C(s) = KD(s) \Rightarrow \quad \frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)H(s)} = \frac{C(s)G(s)}{1 + KL(s)}$$

Writing the loop gain as $KL(s)$ we are interested in tracking the closed-loop poles as "gain" K varies

Root Locus

Characteristic Equation:

$$1 + KL(s) = 0$$

The roots (zeros) of the characteristic equation are the closed-loop poles of the feedback system!!!

The closed-loop poles are a function of the "gain" K

Writing the loop gain as

$$L(s) = \frac{b(s)}{a(s)} = \frac{s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

The closed loop poles are given indistinctly by the solution of:

$$1 + KL(s) = 0, \quad 1 + K\frac{b(s)}{a(s)} = 0, \quad a(s) + Kb(s) = 0, \quad L(s) = -\frac{1}{K}$$

Root Locus

$$\text{RL} = \text{zeros}\{1 + KL(s)\} = \text{roots}\{\text{den}(L) + K\text{num}(L)\}$$

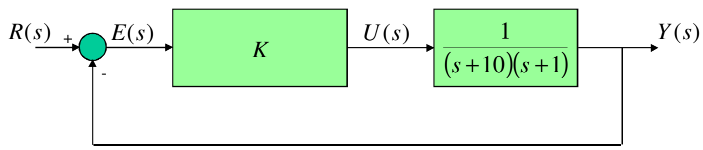
when K varies from 0 to ∞ (positive Root Locus) or from 0 to -∞ (negative Root Locus)

$$K > 0 : L(s) = -\frac{1}{K} \Leftrightarrow \begin{cases} |L(s)| = \frac{1}{K} & \text{Magnitude condition} \\ \angle L(s) = 180^\circ & \text{Phase condition} \end{cases}$$

$$K < 0 : L(s) = -\frac{1}{K} \Leftrightarrow \begin{cases} |L(s)| = -\frac{1}{K} & \text{Magnitude condition} \\ \angle L(s) = 0^\circ & \text{Phase condition} \end{cases}$$

Root Locus by Characteristic Equation Solution

Example:



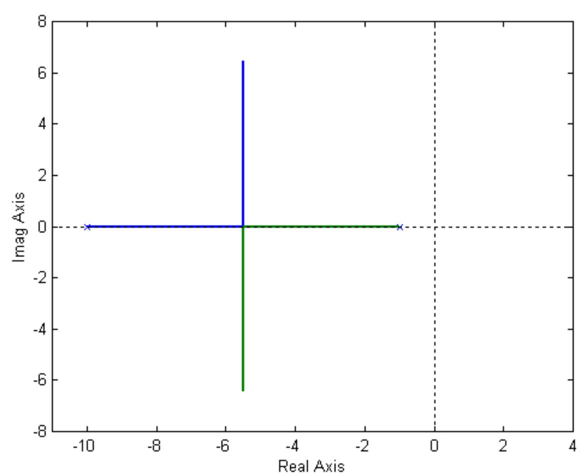
$$\frac{Y(s)}{R(s)} = \frac{K}{s^2 + 11s + (10+K)}$$

Closed-loop poles: $1 + L(s) = 0 \Leftrightarrow s^2 + 11s + (10+K) = 0$

$$s = -5.5 \pm \frac{\sqrt{81-4K}}{2}$$

$s = -1, -10$	$K = 0$
$s = -5.5 \pm \frac{\sqrt{81-4K}}{2}$	$81 - 4K > 0$
$s = -5.5$	$81 - 4K = 0$
$s = -5.5 \pm i\frac{\sqrt{4K-81}}{2}$	$81 - 4K < 0$

Root Locus by Characteristic Equation Solution



We need a systematic approach to plot the closed-loop poles as function of the gain $K \rightarrow$ ROOT LOCUS

Phase and Magnitude of a Transfer Function

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

$$G(s) = K \frac{(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)}$$

The factors K, $(s\text{-}z_j)$ and $(s\text{-}p_k)$ are complex numbers:

$$(s - z_j) = r_j^z e^{i\phi_j^z}, \qquad j = 1\ldots m$$

$$(s - p_k) = r_k^p e^{i\phi_k^p}, \qquad k = 1\cdots p$$

$$K = |K| e^{i\phi^K}$$

$$G(s) = |K| e^{i\phi^K} \frac{r_1^z e^{i\phi_1^z} r_2^z e^{i\phi_2^z} \cdots r_m^z e^{i\phi_m^z}}{r_1^p e^{i\phi_1^p} r_2^p e^{i\phi_2^p} \cdots r_n^p e^{i\phi_n^p}}$$

Phase and Magnitude of a Transfer Function

$$G(s) = |K| e^{i\phi^K} \frac{r_1^z e^{i\phi_1^z} r_2^z e^{i\phi_2^z} \cdots r_m^z e^{i\phi_m^z}}{r_1^p e^{i\phi_1^p} r_2^p e^{i\phi_2^p} \cdots r_n^p e^{i\phi_n^p}}$$

$$= |K| e^{i\phi^K} \frac{r_1^z r_2^z \cdots r_m^z e^{i\left(\phi_1^z + \phi_2^z + \cdots + \phi_m^z\right)}}{r_1^p r_2^p \cdots r_n^p e^{i\left(\phi_1^p + \phi_2^p + \cdots + \phi_n^p\right)}}$$

$$= |K| \frac{r_1^z r_2^z \cdots r_m^z}{r_1^p r_2^p \cdots r_n^p} e^{i\left[\phi^K + \left(\phi_1^z + \phi_2^z + \cdots + \phi_m^z\right) - \left(\phi_1^p + \phi_2^p + \cdots + \phi_n^p\right)\right]}$$

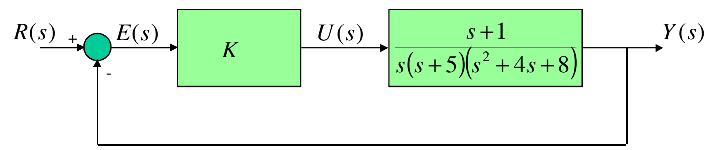
Now it is easy to give the phase and magnitude of the transfer function:

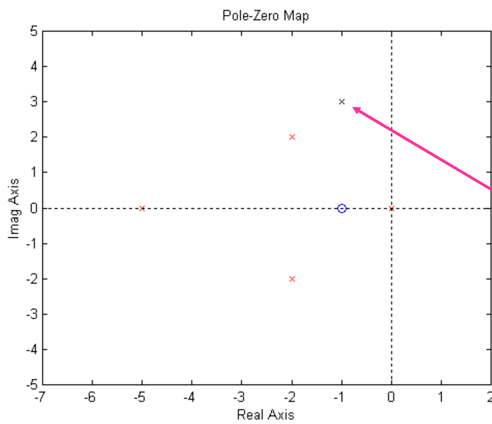
$$|G(s)| = |K| \frac{r_1^z r_2^z \cdots r_m^z}{r_1^p r_2^p \cdots r_n^p},$$

$$\angle G(s) = \phi^K + \left(\phi_1^z + \phi_2^z + \cdots + \phi_m^z\right) - \left(\phi_1^p + \phi_2^p + \cdots + \phi_n^p\right)$$

Root Locus by Phase Condition

Example:





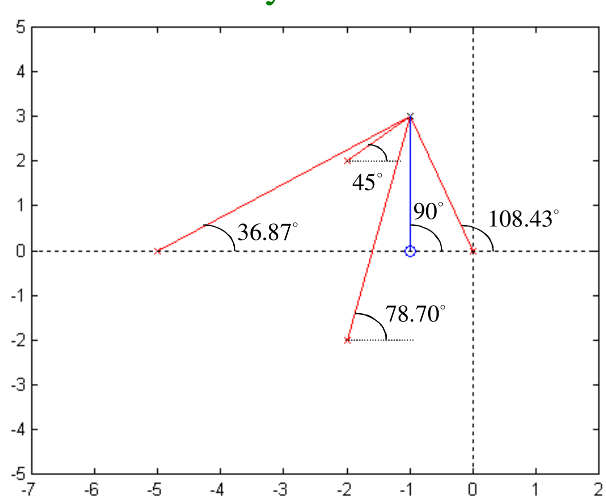
$$L(s) = \frac{s+1}{s(s+5)(s^2+4s+8)}$$

$$= \frac{s+1}{s(s+5)(s+2+2i)(s+2-2i)}$$

$$s_o = -1+3i$$

belongs to the locus?

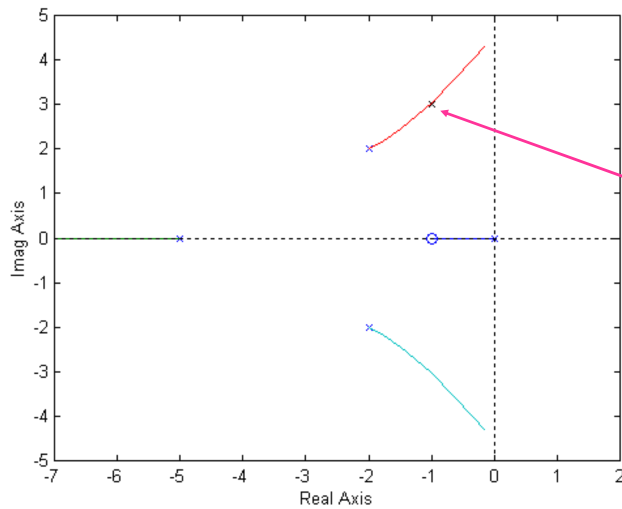
Root Locus by Phase Condition



$90^\circ - [108.43^\circ + 36.87^\circ + 45^\circ + 78.70^\circ] \approx -180^\circ \quad \Rightarrow s_o = -1+3i$ belongs to the locus!

Note: Check code rlocus_phasecondition.m

Root Locus by Phase Condition



$s_o = -1 + 3i$

We need a systematic approach to plot the closed-loop poles as function of the gain $K \rightarrow$ ROOT LOCUS