EACH SECOND COUNTABLE ABELIAN GROUP IS A SUBGROUP OF A SECOND COUNTABLE DIVISIBLE GROUP

TARAS BANAKH AND LIUBOMYR ZDOMSKYĬ

ABSTRACT. It is shown that each pseudonorm $|\cdot|_H$ defined on a subgroup H of an abelian group G can be extended to a pseudonorm $|\cdot|_G$ on G such that the densities of the pseudometrizable topological groups $(H, |\cdot|_H)$ and $(G, |\cdot|_G)$ coincide. We derive from this that any Hausdorff ω-bounded group topology on H can be extended to a Hausdorff ω-bounded group topology on G. In its turn this result implies that each separable metrizable abelian group H is a subgroup of a separable metrizable divisible group G. This result essentially relies on the Axiom of Choice and is not true under the Axiom of Determinacy (which contradicts to the Axiom of Choice but implies the Countable Axiom of Choice).

This paper was motivated by the following question having its origin in functional analysis (see [PZ], [BRZ]): *Is it true that every metrizable separable abelian topological group with no torsion is a subgroup of a metrizable separable divisible abelian group with no torsion?*

From now on all groups considered in the paper are commutative. We recall that a group G is *divisible* (resp. *has no torsion*) if for any element $a \in G$ and a positive integer n the equation $nx = a$ has a solution $x \in G$ (resp. does not have two distinct solutions in G). According to the Baer Theorem [F, 21.1] each divisible group G is *injective* in the sense that each homomorphism $h : B \to G$ defined on a subgroup B of a group A can be extended to a homomorphism $\bar{h} : A \to G$. A classical result of the theory of infinite abelian groups [F, 24.1] asserts that each group (with no torsion) is a subgroup of a divisible group (with no torsion). This result allows us to reduce the above question to the following one: *Can every separable group topology on a subgroup H of a group G be extended to a separable group topology on G?*

Note that without the separability requirement this problem is trivial: just announce H to be an open subgroup of G and take the neighborhood base at the origin of H for a neighborhood base at the origin in the group G. However if the quotient group G/H is uncountable such an extension leads to an unseparable topology on G. So, another less direct approach should be developed.

A classical result in the theory of topological groups asserts that each group topology is generated by a family of continuous pseudonorms, see [Tk, §2]. This observation allows us to reduce the problem of extending group topologies to the problem of extending pseudonorms. As usual, under a *(continuous) pseudonorm* of a (topological) group G we understand a (continuous) non-negative function $|\cdot| : G \to [0, \infty)$ such that $|0| = 0$ and $|x - y| \le |x| + |y|$ for all $x, y \in G$. A pseudonorm $|\cdot|$ is a *norm* provided $|x| = 0$ implies $x = 0$. Each pseudonorm $|\cdot|$ on a group G generates a group topology on G whose neighborhood base at the origin consists of the ε-balls $B_{|\cdot|}(\varepsilon) = \{x \in G : |x| < \varepsilon\}$, $\varepsilon > 0$. The group G endowed with this topology turns into a topological group denoted by $(G, |\cdot|)$.

1991 *Mathematics Subject Classification.* 03E25, 03E60, 20K35, 20K45, 22A05, 54A25, 54A35, 54H05, 54H11.

Key words and phrases. abelian group, divisible group, pseudonorm, second countable group, boundedness index, extension of topologies, Polish group, analytic space, Axiom of Choice, Axiom of Determinacy.

Given a topological space X by $d(X)$ we denote its *density* (that is the smallest size of a dense subset of X), by $w(X)$ its *weight* (that is the smallest size of a base of the topology of X) and by $\chi(X)$ its *character* (i.e., a smallest cardinal τ such that any point $x \in X$ possesses a neighborhood base $\mathcal{B}$ of size $|\mathcal{B}| \le \tau$). It is known that $d(X) = w(X)$ for any (pseudo)metrizable topological space.

Now we are able to formulate the main result of this paper.

Theorem 1. *Any pseudonorm $|\cdot|_H$ defined on a subgroup H of an abelian group G can be extended to a pseudonorm $|\cdot|_G$ on G so that $d(H, |\cdot|_H) = d(G, |\cdot|_G)$.*

Because of its technical character we postpone the proof of this theorem till the end of the paper. Now we consider some its corollaries.

According to [Tk, 4.1], $w(G) = \chi(G) \cdot ib(G)$ for any topological group G where $ib(G)$ stands for the *boundedness index* of G, equal to the smallest cardinal τ such that for any neighborhood U of the origin of G there is a subset $F \subset G$ with $G = F \cdot U$ and $|F| \le \tau$, see [Tk, §3]. Topological groups G with $ib(G) \le \aleph_0$ are called *ω-bounded*, see [Gu] or [Tk]. It is known that a metrizable topological group is ω-bounded if and only if it is separable. Unlike to separable groups, the class of ω-bounded groups is closed under many operations, in particular taking subgroups and Tychonov products, see [Tk] or [Gu].

Taking into account that $|X| \le 2^{w(X)}$ for any Hausdorff topological space X [En, 1.5.1] and $w(G) = \chi(G) \cdot ib(G)$ for any topological group G, we get $|G| \le 2^{\chi(G)}$ for any Hausdorff ω-bounded topological group G. This inequality can be rewritten as $\chi(G) \ge \log |G|$, where $\log \kappa = \min\{\tau : k \le 2^\tau\}$ for a cardinal κ.

Theorem 2. *Any Hausdorff group topology τ_H defined on a subgroup H of an abelian group G can be extended to a Hausdorff group topology τ_G on G so that $ib(G, \tau_G) = ib(H, \tau_H)$, $\chi(G, \tau_G) = \max\{\chi(H, \tau_H), \log |G|\}$ and $w(G, \tau_G) = \max\{w(H, \tau_H), \log |G|\}$.*

In an obvious way Theorem 2 implies

Corollary 1. *Any separable metrizable topology defined on a subgroup H of an abelian group G with $|G| \le \mathfrak{c}$ can be extended to a separable metrizable topology on G.*

Here $\mathfrak{c}$ stands for the size of continuum. The next our corollary follows from Theorem 2 and Theorem 24.1 of [F] asserting that each abelian group H (with no torsion) is a subgroup of a divisible group G (with no torsion) such that $|G| = |H|$.

Corollary 2. *Any Hausdorff topological abelian group H (with no torsion) is a subgroup of a Hausdorff abelian divisible group G (with no torsion) such that $w(G) = w(H)$, $\chi(G) = \chi(H)$ and $ib(G) = ib(H)$.*

The following particular case of the above corollary gives a positive answer to the question stated at the beginning of the paper.

Corollary 3. *Each separable metrizable abelian group H (with no torsion) is a subgroup of a separable metrizable divisible group G (having no torsion).*

In fact, the construction of such a divisible group $G \supset H$ hardly uses Axiom of Choice (see Remark 1). As a result the group G has a complex descriptive structure. We shall show that in general the group G is not analytic. Let us recall that a topological space is *analytic* if it is a metrizable continuous image of a Polish space. As usual, under a *Polish space* we understand a topological space homeomorphic to a separable complete metric space. A topological group is *Polish* (*analytic*) if its underlying topological space is Polish (analytic).

The well-known Open Mapping Principle for Banach spaces generalizes to topological groups as follows: *Any continuous group homomorphism from an analytic group onto a*

Polish group is open. The proof of this Open Mapping Principle follows from Theorem 9.10 [Ke] asserting that any homomorphism $h : G \to H$ from a Polish group G into a ω-bounded group H is continuous provided h has the Baire Property and Theorem 29.5 of [Ke] asserting that analytic subspaces of Polish spaces have Baire Property. We remind that a subset A of a topological space X has *the Baire property* if A contains a G_δ-subset G of X such that $A \setminus G$ is meager in X.

For a group H with no torsion and a positive integer n let $nH = \{ny : y \in H\} \subset H$ and $\frac{1}{n} : nH \to H$ be the map assigning to each element $x \in nH$ a unique $y \in H$ such that $ny = x$.

Proposition 1. *If a Polish group H is a subgroup of a divisible analytic group G with no torsion, then for every positive integer n the map $\frac{1}{n} : nH \to H$ is continuous.*

Proof. The subgroup H, being complete, is closed in G. Then the subgroup $\frac{1}{n}H = \{g \in G : ng \in H\}$, being the preimage of H under the continuous map $n : G \to G$, $n : x \mapsto nx$, is a closed subset of G and thus is analytic. Since the group G is divisible and has no torsion, the map $n : \frac{1}{n}H \to H$, $n : x \to nx$, is a bijective continuous group homomorphism from the analytic group $\frac{1}{n}H$ onto the Polish group H. Applying the Open Mapping Principle for topological groups we conclude that this map is a topological isomorphism and hence the map $\frac{1}{n} : H \to \frac{1}{n}H$ is continuous. Since $nH \subset H$, the map $\frac{1}{n} : nH \to H$ is continuous too. □

Finally we give an example of a Polish group without torsion admitting no embedding into a divisible analytic group without torsion.

Example 1. *There is a Polish group H without torsion such that the map $\frac{1}{2} : 2H \to H$ is discontinuous. This group H cannot be a subgroup of a divisible analytic group with no torsion.*

Proof. For every $k \in \mathbb{N}$ let H_k be a copy of the group $\mathbb{R}$ of reals and let $e_k = 1 \in H_k$. Endow the group H_k with the norm $|x|_k = \sqrt{(\cos(\pi z) - 1)^2 + \sin^2(\pi z) + (2^{-(k+1)}x)^2}$ (which is generated by the usual Euclidean distance under a suitable winding of $H_k = \mathbb{R}$ around a cylinder in $\mathbb{R}^3$). It is easy to verify that $|e_k|_k > 2$ while $|2e_k|_k = 2^{-k}$.

On the direct sum $\oplus_{k\in\mathbb{N}} H_k$ consider the norm $|(x_k)_{k\in\mathbb{N}}| = \sum_{i\in\mathbb{N}} |x_k|_k$ and let H be the completion of $\oplus_{k\in\mathbb{N}} H_k$ with respect to this norm. Then H is a Polish group. We claim that H has no torsion.

Consider the identity inclusion $i : \oplus_{k\in\mathbb{N}} H_k \to \prod_{k\in\mathbb{N}} H_k$ from the direct sum into the direct product endowed with the Tychonov topology. Observe that this direct product is a complete group. To show that the group H has no torsion, it suffices to verify that the extension $\bar{i} : H \to \prod_{k\in\mathbb{N}} H_k$ of the homomorphism i onto the completion H is injective.

It will be convenient to think of elements of the groups $\oplus_{k\in\mathbb{N}} H_k$ and $\prod_{k\in\mathbb{N}} H_k$ as functions $f : \mathbb{N} \to \bigcup_{k\in\mathbb{N}} H_k$.

Assuming that the homomorphism $\bar{i}$ is not injective, we could find an element $f_\infty \in H$ such that $f_\infty \neq 0$ but $\bar{i}(f_\infty) = 0$. Fix any $\varepsilon > 0$ with $\varepsilon < |f_\infty|$. Choose a sequence $(f_n)_{n\in\mathbb{N}} \in \oplus_{k\in\mathbb{N}} H_k$ converging to f_∞ in H. We can assume that $|f_n| > \varepsilon$ for every $n \in \mathbb{N}$. By the continuity of the map $\bar{i}$, we conclude that the sequence $\{i(f_n)\}_{n\in\mathbb{N}}$ converges to zero in $\prod_{k\in\mathbb{N}} H_k$ (this means that the function sequence (f_n) is pointwise convergent to zero). Since the sequence (f_n) is Cauchy in H, there is $m \in \mathbb{N}$ such that $|f_m - f_j| < \frac{\varepsilon}{2}$ for any $j \geq m$. Without loss of generality, we can assume that $f_m(k) = 0$ for all $k > m$. Since for every k $\lim_{j\to\infty} f_j(k) = 0$, we can find $j > m$ so large that $|f_j(k)|_k < \frac{\varepsilon}{2m}$ for all $k \leq m$. Then $|f_m - f_j| = \sum_{k=1}^{\infty} |f_m(k) - f_j(k)|_k \geq \sum_{k=1}^{m} |f_m(k) - f_j(k)|_k \geq$

$\sum_{k=1}^m |f_m(k)|_k - \sum_{k=1}^m |f_j(k)|_k \geq |f_m| - \sum_{k=1}^m \frac{\varepsilon}{2m} > \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}$, which contradicts to $|f_m - f_j| < \frac{\varepsilon}{2}$.

Therefore, the homomorphism $\bar{i}$ is injective and the group H has no torsion. Since $|e_k| = |e_k|_k > 2$ and $|2e_k| = |2e_k|_k = 2^{-k}$ for all k, we see that the sequence $(2e_k)$ converges to zero in H while (e_k) does not. This means that the map $\frac{1}{2} : 2H \to H$ is discontinuous. □

.

Remark 1. Corollary 3 can not be proven without the full Axiom of Choice and is not true under the Axiom of Determinacy. This axiom contradicts the Axiom of Choice but implies its weaker form, the Countable Axiom of Choice, see [JW, §9.2 and §9.3]. It is known that under the Axiom of Determinacy, any subset of a Polish space has the Baire Property, see [Ke, 8.35]. This fact and Theorem 9.10 of [Ke] implies that under Axiom of Determinacy the Open Mapping Principle for topological groups holds in the following more strong form: any continuous homomorphism $h : H \to G$ from a ω-bounded group H onto a Polish group G is open. Using this stronger form of the Open Mapping Principle and repeating the proof of Proposition 1 we see that under the Axiom of Determinacy this proposition holds without the analycity assumption on the group G. Thus we come to a rather unexpected conclusion: *Under the Axiom of Determinacy the group H from Example 1 cannot be embedded into a metrizable separable divisible group with no torsion, in spite of the fact that algebraically, H is a subgroup of the countable product $\mathbb{R}^\omega$ of lines.* This shows that Corollary 3 is not true under the Axiom of Determinacy.

1. Proof of Theorem 1

In the proof of Theorem 1 we shall need one combinatorial lemma. A collection $\mathcal{A}$ of subsets of a set X is called *k-uniform* where $k \in \mathbb{N}$ if $|A| = k$ for each $A \in \mathcal{A}$; $\mathcal{A}$ is *disjoint* if it consists of pairwise disjoint sets.

Lemma 1. *Suppose $k \in \mathbb{N}$ and $\mathcal{A}$, $\mathcal{B}$ are two disjoint k-uniform finite collections of subsets of an infinite set X. Then there is a subset $I \subset X$ such that $|I \cap C| = 1$ for each $C \in \mathcal{A} \cup \mathcal{B}$.*

Proof. It is easy to construct k-uniform disjoint finite collections $\mathcal{C}$, $\mathcal{D}$ of subsets of X such that $\mathcal{A} \subset \mathcal{C}$, $\mathcal{B} \subset \mathcal{D}$, $|\mathcal{C}| = |\mathcal{D}|$, and $\cup\mathcal{C} = \cup\mathcal{D}$. Let $n = |\mathcal{C}| = |\mathcal{D}|$ and write $\mathcal{C} = \{C_1, \dots, C_n\}$, $\mathcal{D} = \{D_1, \dots, D_n\}$. Consider the matrix $[a_{ij}]_{i,j=1}^n$ where $a_{ij} = \frac{1}{k}|C_i \cap D_j|$ and observe that it is *double stochastic*, that is $\sum_{i=1}^n a_{ij} = 1 = \sum_{j=1}^n a_{ij}$ for all $i, j \in \{1, \dots, n\}$. According to the Birkhoff Theorem (see [Bi], [Ga, p.556], or [A, 8.40]) each double stochastic matrix is a convex combination of *permutating matrices*, that is matrices of the form $[\delta_{i,\sigma(j)}]_{i,j=1}^n$ where σ is a permutation of the set $\{1, \dots, n\}$ and $[\delta_{ij}]_{i,j=1}^n$ is the identity matrix. This result implies the existence of a permutation σ of the set $\{1, \dots, n\}$ such that $a_{i,\sigma(i)} > 0$ for all i. This means that the intersection $C_i \cap D_{\sigma(i)}$ is not empty and thus contains some point x_i. Let $I = \{x_1, \dots, x_n\}$ and observe that $|C \cap I| = 1$ for any element $C \in \mathcal{C} \cup \mathcal{D} \supset \mathcal{A} \cup \mathcal{B}$. □

Theorem 1 will be proved by induction whose inductive step is based of the following

Lemma 2. *Let H be a subgroup of a group G such that $pG \subset H$ for some prime number p. Then any pseudonorm $|\cdot|_H$ on H can be extended to a pseudonorm $|\cdot|_G$ on G so that $d(G, |\cdot|_G) = d(H, |\cdot|_H)$.*

Proof. The quotient group G/H has prime exponent p and thus has a basis which can be written as $\{g^\alpha + H : \alpha < \mu\}$ for some ordinal μ, see [F, 16.4]. It will be convenient to complete this basis by zero letting $g^\mu = 0$. It follows that any element of G can be represented in the form $x = u + \sum_{i\in\omega} g^{\alpha_i}$ where $u \in H$ and the set $\{i \in \omega : \alpha_i \neq \mu\}$ is finite. For any element $x \in G$ let $\mathrm{Rep}(x) = \{(u, (\alpha_i)_{i\in\omega}) \in H\times[0, \mu]^\omega : x = u+\sum_{i\in\omega} g^{\alpha_i}\}$. Observe that for any $x \in H$ and $(u, (\alpha_i)_{i\in\omega}) \in \mathrm{Rep}(x)$ the number p divides the cardinality of the set $\{i \in \omega : \alpha_i = \alpha\}$ for each ordinal $\alpha < \mu$.

Let $|\cdot|_H$ be a pseudonorm on H. Define a function $\rho : G \times G \to [0, \infty)$ letting

$$\rho(x, y) = \inf\{|u - v|_H + \sum_{i\in\omega} |pg^{\alpha_i} - pg^{\beta_i}|_H : (u, (\alpha_i)_{i\in\omega}) \in \mathrm{Rep}(x), (v, (\beta_i)_{i\in\omega}) \in \mathrm{Rep}(y)\}$$

for $x, y \in G$.

It is easy to see that ρ is an invariant pseudometric on G. Let us show that $d(G, \rho) \leq d(H, |\cdot|_H)$. Let D be a dense subset of $(H, |\cdot|_H)$ with $|D| = d(H, |\cdot|_H)$ and $I \subset [0, \mu]$ be a subset of size $|I| \leq d(H, |\cdot|_H)$ such that $I \ni \mu$ and the set $\{pg^\alpha : \alpha \in I\}$ is dense in the subspace $\{pg^\alpha : \alpha < \mu\}$ of $(H, |\cdot|_H)$. Then the set $E = \{x \in G : \mathrm{Rep}(x) \cap (D \times I^\omega) \neq \emptyset\}$ is a dense subset of (G, ρ) with $|E| \leq d(H, |\cdot|_H)$. This proves the inequality $d(G, \rho) \leq d(H, |\cdot|_H)$.

It remains to show that $\rho(x, y) = |x-y|_H$ for any $x, y \in H$. Fix arbitrary $(u, (\alpha_i)_{i\in\omega}) \in \mathrm{Rep}(x)$, $(v, (\beta_i)_{i\in\omega}) \in \mathrm{Rep}(y)$. For every $\alpha < \mu$ let $A(\alpha) = \{i \in \omega : \alpha_i = \alpha\}$ and $B(\alpha) = \{i \in \omega : \beta_i = \alpha\}$. Since $x, y \in H$, the number p divides the cardinalities of the sets $A(\alpha)$, $B(\alpha)$ for all ordinals $\alpha < \mu$. Applying Lemma 1, find a subset $I \subset \omega$ such that $|C \cap I| = \frac{1}{p}|C|$ for every nonempty subset $C \in \{A(\alpha), B(\alpha) : \alpha < \mu\}$. Then

$$\begin{aligned} x = \ & u + \textstyle\sum_{i\in\omega} g^{\alpha_i} = \ u + \sum_{\alpha<\mu} |A(\alpha)| \cdot g^\alpha = u + \sum_{\alpha<\mu} p|I \cap A(\alpha)|\, g^\alpha = \\ & = u + \sum_{\alpha<\mu} \sum_{i\in I\cap A(\alpha)} pg^{\alpha_i} = u + \sum_{i\in I} pg^{\alpha_i}. \end{aligned}$$

By analogy, $y = v + \sum_{i\in I} pg^{\beta_i}$. Consequently,

$$\begin{aligned} |u - v|_H + \ & \textstyle\sum_{i\in\omega} |pg^{\alpha_i} - pg^{\beta_i}|_H \geq |u - v|_H + \sum_{i\in I} |pg^{\alpha_i} - pg^{\beta_i}|_H \geq \\ & \geq |(u + \textstyle\sum_{i\in I} pg^{\alpha_i}) - (v + \sum_{i\in I} pg^{\beta_i})|_H = |x - y|_H \end{aligned}$$

Passing to the infimum, we get $\rho(x, y) \geq |x - y|_H$. The proof of the inverse inequality is straightforward, hence $\rho(x, y) = |x - y|_H$. Letting $|x|_G = \rho(x, 0)$ for $x \in G$ we define a pseudonorm on G extending the pseudonorm $|\cdot|_H$ so that $d(G, |\cdot|_G) = d(H, |\cdot|_H)$. □

Lemma 3. *Let H be a subgroup of a group G such that the quotient group G/H is periodic. Then any pseudonorm $|\cdot|_H$ on H can be extended to a pseudonorm $|\cdot|_G$ on G so that $d(G, |\cdot|_G) = d(H, |\cdot|_H)$.*

Proof. Let $(p_i)_{i=1}^\infty$ be a sequence of prime numbers such that for every prime number p the set $\{i \in \mathbb{N} : p_i = p\}$ is infinite. Let $H_0 = H$ and for $i \geq 0$ let $H_{i+1} = \frac{1}{p_{i+1}} H_i = \{x \in G : p_{i+1}x \in H_i\}$. Because of the periodicity of the quotient group G/H we get $G = \bigcup_{i=1}^\infty H_i$.

Let $|\cdot|_H$ be any pseudonorm on H and D_0 be a dense subset of the topological group $(H, |\cdot|_H)$ with $|D_0| = d(H, |\cdot|_H)$. Let $|\cdot|_0 = |\cdot|_H$. Using the previous lemma, by induction for every $i \geq 1$ find a pseudonorm $|\cdot|_i$ on the group H_i and a dense subset D_i of the topological group $(H_i, |\cdot|_i)$ such that $|x|_i = |x|_{i-1}$ for each $x \in H_{i-1}$ and $|D_i| = |D_{i-1}|$.

Completing the inductive construction, define a pseudonorm $|\cdot|_G$ on the group G letting $|x|_G = |x|_i$ where $x \in H_i$. It is clear that $|\cdot|_G$ extends $|\cdot|_H$ and $D = \bigcup_{i=1}^\infty D_i$ is a dense set in the topological group $(G, |\cdot|_G)$ with $|D| = |D_0| = d(H, |\cdot|_H)$. This yields $d(G, |\cdot|_G) \leq d(H, |\cdot|_H)$. □

Finally we are able to complete the

Proof of Theorem 1. Let H be a subgroup of a group G and $|\cdot|_H$ be a pseudonorm on H. According to [F, 24.1] the group H is a subgroup of a divisible group E. Moreover, according to Lemma 24.3 [F] we can assume that the quotient group E/H is periodic. Applying the previous lemma, extend the pseudonorm $|\cdot|_H$ to a pseudonorm $|\cdot|_E$ on E so that $d(E,|\cdot|_E) = d(H,|\cdot|_H)$. According to Baer Theorem [F, 21.1], each divisible group is injective. Consequently, there is a group homomorphism $h : G \to E$ extending the identity map $H \to H \subset E$. Define a pseudonorm $|\cdot|_G$ on G letting $|x|_G = |h(x)|_E$ for $x \in G$ and observe that $|\cdot|_G$ extend $|\cdot|_H$ and $d(H,|\cdot|_H) \le d(G,|\cdot|_G) \le d(E,|\cdot|_E) = d(H,|\cdot|_H)$. $\square$

2. Proof of Theorem 2

Let H be a subgroup of a group G and τ_H be a Hausdorff group topology on H.

First we define a Hausdorff group topology $\tau_{G/H}$ on the quotient group G/H such that $ib(G/H, \tau_{G/H}) \le \omega$ and $\chi(G/H, \tau_{G/H}) \le \log|G/H|$. By [F, 24.1] G/H is a subgroup of a divisible group E with $|E| = |G/H|$. Applying Theorem 23.1 [F] (on the structure of divisible groups), we can show that the group E is isomorphic to a subgroup of the power $\mathbb{T}^\kappa$ of the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ where $\kappa = \log|G/H| \le \log|G|$. Observe that $\mathbb{T}^\kappa$ endowed with the natural Tychonov product topology is a compact topological group with $ib(\mathbb{T}^\kappa) = \omega$ and $\chi(\mathbb{T}^\kappa) = \kappa \le \log|G|$.

Consequently, the group G/H, being isomorphic to a subgroup of $\mathbb{T}^\kappa$, carries a Hausdorff group topology $\tau_{G/H}$ such that $ib(G/H, \tau_{G/H}) \le \omega$ and $\chi(G/H, \tau_{G/H}) \le \kappa \le \log|G|$.

Fix a neighborhood base $\mathcal{B}$ of size $|\mathcal{B}| = \chi(H, \tau_H)$ at the origin of the topological group (H, τ_H). Applying [Tk, 2.3], for every $U \in \mathcal{B}$ fix a continuous pseudonorm $|\cdot|_U$ on H such that $\{x \in H : |x|_U < 1\} \subset U$. By Theorem 1, the pseudonorm $|\cdot|_U$ can be extended to a pseudonorm $\|\cdot\|_U$ on G such that $d(G, \|\cdot\|_U) = d(H, |\cdot|_U)$. The continuity of the identity map $(H, \tau_H) \to (H, |\cdot|_U)$ implies that $ib(H, |\cdot|_H) \le ib(H, \tau_H)$, see [Tk, 3.2]. Since the density and the boundedness index coincide for (pseudo)metrizable topological groups [Tk, §3], we conclude that $ib(G, \|\cdot\|_U) = d(G, \|\cdot\|_U) = d(H, |\cdot|_U) = ib(H, |\cdot|_U) \le ib(H, \tau_H)$.

Let τ_G be the smallest topology on G making continuous the quotient homomorphism $(G, \tau_G) \to (G/H, \tau_{G/H})$ and the identity map $(G, \tau) \to (G, \|\cdot\|_U)$ for all $U \in \mathcal{B}$. It is easy to see that τ_G is a Hausdorff group topology on G inducing the topology τ_H on the subgroup H.

Observe that the topological group (G, τ_G) can be identified with a subgroup of the product $G/H \times \prod_{U \in \mathcal{B}} (G, \|\cdot\|_U)$ of topological groups whose boundedness indices do not exceed $ib(H, \tau_H)$ and characters do not exceed $\log|G|$. According to [Tk, 3.2], the boundedness index of such a product does dot exceed $ib(H, \tau_H)$ while its character does not exceed $\chi(G/H) \cdot |\mathcal{B}| \le \log|G| \cdot \chi(H, \tau_H)$. Consequently, $ib(G, \tau_G) \le ib(H, \tau_H)$, $\chi(G, \tau_G) \le \chi(H, \tau_H) \cdot \log|G|$, and $w(G, \tau_G) = \chi(G, \tau_G) \cdot ib(G, \tau_G) \le \log|G| \cdot \chi(H, \tau_H) \cdot ib(H, \tau_H) = w(H, \tau_H) \cdot \log|G|$.

Acknowledgement. The authors express their sincere thanks to Sasha Ravsky and Igor Protasov for valuable and stimulating discussions (concerning stochastic matrices).

References

[A] M. Aigner. Combinatorial Theory. – Springer-Verlag, 1979.

[BRZ] T. Banakh, A. Plichko, A. Zagorodnyuk. Automatic continuity of polynomial operators between topological abelian groups (in preparation).

[Bi] G. Birkhoff, Tres observaciones sobre el algebra lineal // Rev. Univ. Nac Tucumaán. ser. A. – 1946. – V.5. –P.147-151.

[En] R. Engelking. *General Topology.* – Warszawa: PWN, 1985.

[F] L. Fuchs. *Infinite Abelian Groups, I.* – New York: Academic Press, 1970.

[Ga] F.R. Gantmacher. *Matrix Theory.* – Moscow: Nauka, 1967 (in Russian). P.556.

[Gu] I. Guran, On topological groups close to being Lindelöf, *Soviet Math. Dokl.* – 1981. – V.23. – P.173–175.

[JW] W. Just, M. Weese, *Discovering Modern Set Theory, I* (GSM, Vol.8. – Providence, RI: Am. Math. Soc., 1996.

[Ke] A.S. Kechris. *Classical descriptive set theory.* – Berlin: Springer-Verlag, 1995.

[PZ] A. Plichko, A. Zagorodnyuk. Isotropic mappings and automatic continuity of polynomial, analytic, and convex operators // in: *General Topology in Banach Spaces* (T.Banakh ed.). – New York: Nova Publ., 2001. – P.1–13.

[Tk] M. Tkachenko, Introduction to topological groups, *Topology Appl.* – 1998. – V.86. – P.179–231.

E-mail address: `tbanakh@franko.lviv.ua`

Department of Mathematics, Ivan Franko Lviv National University, Universytetska 1, Lviv, 79000, Ukraina