DR. GUIRGUIS YASSA
Robinson Aerial Surveys, Inc.
Newton, New Jersey 07860

Orthogonal Transformations in Triangulation Adjustment*

The Gram-Schmidt process allows the reduction of very large systems of equations to smaller systems through the repeated application of partitioning a vector space into a subspace and its orthogonal complement.

ABSTRACT: *The numerical solution of point estimators as well as interval estimators affects the problem of aerial triangulation adjustment via orthogonal transformations. Two methods are presented that avoid the formation of normal equations. The first method makes use of the Gram-Schmidt orthonormalization process. The second method utilizes the Householder orthogonal transformations. Problems are reported that arose during implementation of the simultaneous adjustment of bundles using Householder transformations, together with their solutions.*

INTRODUCTION

IT HAS BEEN said in the past that aerial triangulation is simply a problem of matrix inversion. To be more optimistic, it can be said that aerial triangulation is a problem of solving a large linear system of equations.

The classical method of solution of such a linear system, which usually has a rectangular coefficient matrix of order $n \times p$ where $n > p$ is based on: 1) the formation of a set of normal equations which have a square coefficient matrix of order $p \times p$; and (2) the solution of such normal equations by a direct or iterative method.

An alternate approach for this least-squares problem is the solution via orthogonal transformations. This is based on the direct manipulation of the columns of the original rectangular linear system, which is usually known as the observation equation. The solution via orthogonal transformations avoids the intermediate step of forming the normal equations which may be ill-conditioned, and hence it is theoretically more stable.

* Presented at the Annual Convention of the American Society of Photogrammetry in Washington, D.C., March 1973.

OBSERVATION EQUATIONS IN THE SIMULTANEOUS ADJUSTMENT OF BUNDLES

The mathematical model for the simultaneous adjustment of bundles in aerial triangulation can be derived from the projective relationships between the photograph and the terrain. (See Schmid, 1959). It can be reduced eventually to a linear statistical model for the form:

$$Y = X\beta + e \tag{1}$$

$$E(e) = 0 \tag{2}$$

$$D(e) = \sigma^2 I \tag{3}$$

where $Y = n \times 1$ random vector derived from the observations, i.e., the measured x- and y-plate coordinates; $\beta = p \times 1$ vector of unknown parameters, such as the spatial coordinates of pass points and the positions and attitudes of the camera during exposures (estimates, $\hat{\beta}$, for β are being sought in the adjustment problem); X = a known $n \times p$ coefficient matrix; $e = n \times 1$ unobservable random vector, which is a function of the measurement errors (an estimate $\hat{e}$ for e is sought in the adjustment); $E(e)$ denotes the expectation of the random vector e, usually assumed equal to the $n \times 1$ zero vector 0; $D(e)$ denotes the dispersion matrix of the random vector e; σ is the standard error of the observations. For details of such reduction, see Yassa & McNair (1973).

To get an idea about the structure of the coefficient matrix, consider a small block of 2×3 photographs, as shown in Figure 1. Assume that four error-free complete ground control points are available at the corners of the block. Assume further that no auxiliary data are observed. Then there are six unknown elements of exterior orientation for each of the six photographs and three unknown spatial coordinates for each of the 11 pass points. Hence, the number of unknowns p in the adjustment of the block would be:

$$p = 6 \times 6 + 3 \times 11 = 69.$$

As for the number of equations n in the adjustment problem, two observation equations can be formed for each measured point on a photograph, one for the x-coordinate and the other for the y-coordinate measurement. As there are six measured points for each end photo in the strip and nine points for each intermediate photo, the number of observation equations n would be:

$$n = 2 \times 2(6 + 9 + 6) = 84.$$

Hence, the coefficient matrix is of the order of 84×69. However, the coefficient matrix is a highly sparse matrix characterized by a high degree of orthogonality between its columns. This orthogonality is due to the fact that, prior to the adjustment, there is no correlation between: (a) the orientation elements of different photographs in the block; (b) the ground coordinates of different pass points in the block; and (c) a photograph and the pass points that are not measured on it. Some correlation is, however, introduced by the adjustment. For a schematic representation of the structure of the coefficient matrix for this sub-block, see Figure 2.

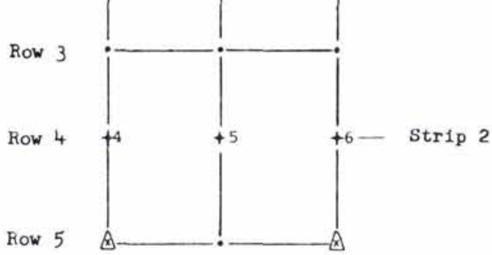

FIG. 1. A sub-block of 2 by 3 photographs. Principal points are indicated with plus symbols, pass points by round dots, and a triangle with an x inside indicates a complete ground control point (or two separate horizontal and vertical points).

Classical Least Squares Solution via Normal Equations

The classical method of solution for the unknown parameters may be briefly summarized as follows:

1. Form the normal equations:

$$X'X\hat{\beta} = X'Y \tag{4}$$

where X' denotes the transpose of the coefficient matrix, $\hat{\beta}$ denotes the least squares estimates for the unknown parameters β. It is to be noted that the resulting coefficient matrix of the normal equations $X'X$, is a positive-definite symmetric matrix.

2. Solve the linear System 4 by a direct or iterative method, to obtain the least squares estimates $\hat{\beta}$ for the unknown parameters.

3. Find estimates for the observation errors e:

$$\hat{e} = Y - X\,\hat{\beta}. \tag{5}$$

4. Find an estimate for the variance of the observation errors:

$$\hat{\sigma}^2 = \hat{e}'\hat{e}/(n-p) \tag{6}$$

where $\hat{e}'$ denotes the transpose of the computed vector of residuals $\hat{e}$.

5. The Covariance matrix of the unknown parameters $\hat{\beta}$, is given by:

$$D(\hat{\beta}) = \sigma^2(X'X)^{-1}, \tag{7}$$

i.e., it is obtained by the inversion of the coefficient matrix of the normal Equations 4.

Least Squares Solution by Orthonormalization of the Coefficient Matrix

The p-columns of the coefficient matrix X, viewed as vectors in R^n span a vector space of R^n denoted by **V**. As these column vectors are linearly independent in the problem of bundle adjustment, they form an arbitrary basis for V. The dimension of V is p.

An orthonormal basis $U^{(p)} = \{u_1, u_2, \ldots u_p\}$ for the vector space V may be obtained by the Gram-Schmidt orthonormalization process which is a recursive p-step procedure as follows:

1. Normalize the first column vector v_1 in X to obtain the orthonormal set $\{u_1\}$ which consists of one vector $u_1 \in R^n$:

$$u_1 = v_1/\,||v_1|| \tag{8}$$

where $||v_1||$ is the Euclidean norm of vector v_1: If the elements of vector v_1 are $v_{11}, v_{21}, \ldots v_{n1}$, then

$$||v_1|| = (v_{11}^2 + v_{21}^2 + \ldots + v_{n1}^2)^{1/2} \tag{9}$$

$u_1 \epsilon R^n$ denotes that the vector u belongs to the vector space R^n.

2. For each step k: $k = 2, 3, \ldots p$, assume there is an orthonormal basis $U^{(k-1)} = \{u_1, u_2, \ldots u_{k-1}\}$ for the first $k - 1$ columns in the coefficient matrix. The orthonormal basis $U^{(k)} = \{u_1, u_2, \ldots u_k\}$ is the union of $U^{(k-1)}$ and u_k where u_k is computed from the k-th column vector v_k as follows:

$$w_k = v_k - \sum_{j=1}^{k-1} <v_k, u_j> u_j \tag{10}$$

$$u_k = w_k / ||w_k|| \tag{11}$$

where $<v_k, u_j>$ denotes the scalar product of the vectors v_k and u_j.

The transition matrix from the arbitrary basis $X = \{v_1, v_2, \ldots v_p\}$ to the orthonormal basis $Q = \{u_1, u_2, \ldots u_p\}$ is an upper triangular matrix R, of full rank p, thus:

$$Q = XR \tag{12}$$

where X and Q are both $n \times p$ matrices, whereas R is a $p \times p$ matrix. As R is of full rank, it is invertible, and

$$X = QR^{-1}. \tag{13}$$

Substituting From 13 into 4, then the point estimators that are classically obtained from the solution of the normal Equations 4, may be directly obtained from:

$$\hat{\beta} = RQ'Y \tag{14}$$

and the intermediate step of forming normal Equations 4 is eliminated.

Thus for the simultaneous adjustment of

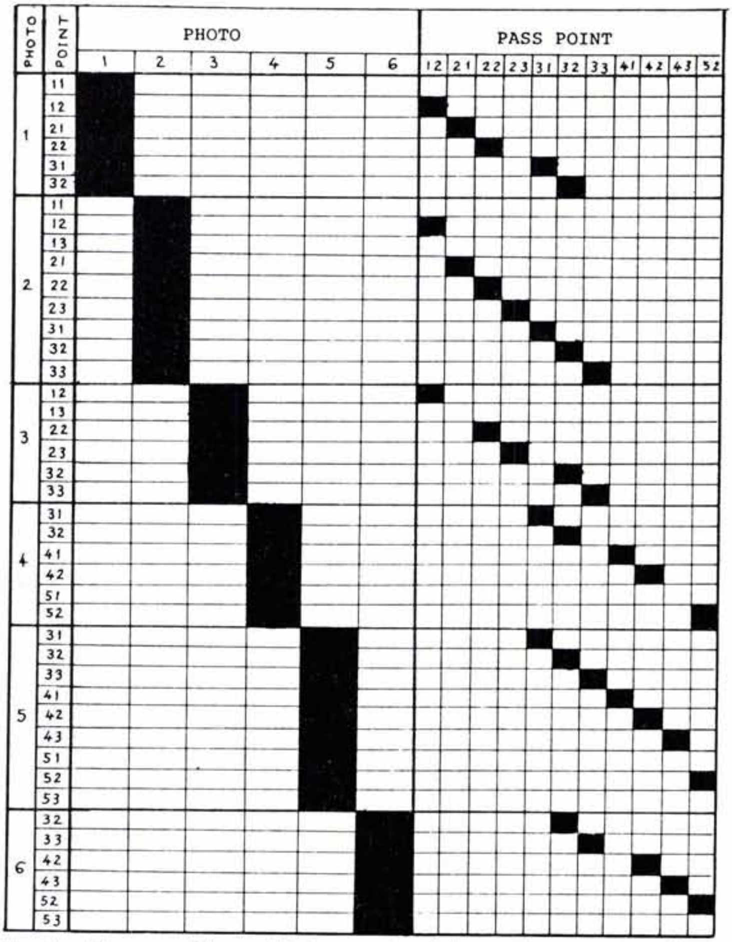

FIG. 2. Diagram of the coefficient matrix of observation equations for a six-photo aerial triangulation. The near-square areas on the right indicate arrays of 2×3 non-zero entries, areas on the left (twice as large) are arrays of 2×6 non-zero entries.

bundles by this method, storage space is needed for:

(a) the random $n \times 1$ vector Y,

(b) the upper triangular matrix R, which is also a sparse matrix, and

(c) the $n \times p$ orthogonal matrix Q. The storage of Q in the main memory of an electronic digital computer would impose a severe limitation on the size of blocks that can be adjusted all at once in the main memory. The adjustment of very large blocks could, however, be made by partitioning the vector space into orthogonal subspaces using auxiliary storage devices. See Yassa (1974).

The random vector of residuals, $\hat{e}$ is simply the component of Y which is orthogonal to all the vectors of the orthonormal basis $U^{(p)}$. Thus $\hat{e}$ is computed from:

$$\hat{e} = Y - \sum_{j=1}^{p} < Y, u_j > u_j \tag{15}$$

i.e., $\hat{e}$ can be computed even before the computation of the unknown parameters $\hat{\beta}$.

The point estimate $\hat{\sigma}^2$ may be computed from:

$$\hat{\sigma}^2 = \hat{e}'\, \hat{e}/(n-p).$$

Applying the law of expectation to Equation 14, one may verify that the estimates $\hat{\beta}$ are unbiased.

The covariance matrix of $\hat{\beta}$ may be also derived from Equation 14:

$$\begin{aligned} D(\hat{\beta}) & \\ &= R\,Q'\,D(Y)\,(RQ')' \\ &= R\,Q'\,\sigma^2 I\,Q\,R' \text{(because } D(Y)=D(e)=\sigma^2 I) \\ &= \sigma^2\,R\,Q'Q\,R' \\ &= \sigma^2\,R\,R' \text{ (because } Q \text{ is orthonormal).} \end{aligned} \tag{16}$$

Thus, the accuracy of the estimates $\hat{\beta}$ derived by Gram-Schmidt orthonormalization process, can be evaluated without any tedious matrix inversion as in the classical method.

Least Squares Solution by Decomposition of the Coefficient Matrix

Let Q be an $n \times n$ orthogonal matrix chosen such that:

$$QX = R = \begin{bmatrix} \tilde{R} \\ O \end{bmatrix} \tag{17}$$

where R is an $n \times p$ matrix with zero entries below the main diagonal, $\tilde{R}$ is a $p \times p$ upper triangular matrix, and O is a $(n-p) \times p$ matrix with zero entries. Apply the orthogonal transformation Q to both sides of the observation Equations 1, and set:

$$C = QY \tag{18}$$

$$\eta = Qe. \tag{19}$$

Then the observational equations reduce to:

$$C = R\beta + \eta \tag{20}$$

$$\text{where } E(\eta) = QE(e) = 0 \tag{21}$$

$$\begin{aligned} \text{and} \quad D(\eta) &= QD(e)Q' \\ &= \sigma^2 QQ' \\ &= \sigma^2 I \text{ (because } Q \text{ is orthogonal)} \end{aligned} \tag{22}$$

and the normal equations reduce to:

$$R'R\hat{\beta} = R'C \tag{23}$$

Let the $n \times 1$ column vector C be partitioned:

$$C = \begin{bmatrix} \dot{C} \\ \ddot{C} \end{bmatrix} \tag{24}$$

where $\dot{C}$ is a $p \times 1$ vector and $\ddot{C}$ is a $(n-p) \times 1$ vector. Then the normal Equations 23 can be written in the form

$$\begin{bmatrix} \tilde{R}' & O' \end{bmatrix} \begin{bmatrix} \tilde{R} \\ O \end{bmatrix} \hat{\beta} = \begin{bmatrix} \tilde{R}' & O' \end{bmatrix} \begin{bmatrix} \dot{C} \\ \ddot{C} \end{bmatrix}. \tag{25}$$

The first p-equations of Equation 25 are:

$$\tilde{R}'\tilde{R}\hat{\beta} = \tilde{R}'\dot{C}. \tag{26}$$

If the rank of the coefficient matrix X is p, it can be shown that $\tilde{R}$ has also a rank p and, hence, it is invertible. Multiply both sides of Equation 26 by $(R')^{-1}$:

$$\tilde{R}\,\hat{\beta} = \dot{C} \tag{27}$$

where $\dot{C}$ is a $p \times 1$ vector representing the first p-elements of the transformed vector QY. This is an upper triangular system, which can be readily solved for $\hat{\beta}$ in a backward scheme, i.e., the unknown parameter $\hat{\beta}_p$ is computed first, $\hat{\beta}_{p-1}$ is computed next, and $\hat{\beta}_1$ is computed last. Thus by applying the orthogonal transformation Q, the intermediate step of forming the normal equations is eliminated.

An unbiased estimate for σ^2 is obtained from

$$\hat{\sigma}^2 = \eta'\eta/(n-p) \tag{28}$$

which can be reduced to the form

$$\hat{\sigma}^2 = \ddot{C}'\ddot{C}/(n-p) \tag{29}$$

where $\ddot{C}$ is an $(n-p) \times 1$ vector representing the last $(n-p)$ elements of the transformed vector QY.

It is worth noting that:

$$R'R = \begin{bmatrix} \tilde{R}' & O' \end{bmatrix} \begin{bmatrix} \tilde{R} \\ O \end{bmatrix} = \tilde{R}'\tilde{R}$$

$$\begin{aligned} \text{and} \quad R'R &= (QX)'\,(QX) = X'Q'\,QX \\ &= X'IX = X'X \\ \text{and hence} \quad X'X &= \tilde{R}'\tilde{R} \end{aligned} \tag{30}$$

i.e., $\tilde{R}'\tilde{R}$ is simply the Choleski decomposition of the coefficient matrix of the normal equations, $X'X$.

The covariance matrix of the estimates $\hat{\beta}$ may be obtained from Equations 7 and 30:

$$D(\hat{\beta}) = \sigma^2 (\tilde{R}'\tilde{R})^{-1}. \quad (31)$$

To realize the triangular decomposition, Golub (1965) advocated the use of Householder orthogonal transformations. Golub's algorithm is a recursive p-step procedure defined by:

$$X^{(1)} = X \quad (32)$$

$$X^{(k+1)} = Q^{(k)} X^{(k)} \quad (k=1,2, \ldots p). \quad (33)$$

In order to get an upper triangular matrix $X^{(p+1)}$, every orthogonal matrix $Q^{(k)}$; $(k=1, 2, \ldots p)$ should transform all elements of the k-th column of $X^{(k)}$ below the main diagonal to zero. This is satisfied by putting:

$$Q^{(k)} = I - \alpha_k U^{(k)} U^{(k)\prime} \quad (34)$$

where

$$\alpha_k = 1/[\sigma_k(\sigma_k + x_{kk}^{(k)})] \quad (35)$$

$$\sigma_k = \pm (\sum_{i=k}^{n} (x_{ik}^{(k)})^2)^{1/2} \text{ with} \begin{cases} + \text{ for } x_{kk}^{(k)} \geq 0 \\ - \text{ for } x_{kk}^{(k)} < 0. \end{cases} \quad (36)$$

I is an $n \times n$ unit matrix, $U^{(k)}$ is an $n \times 1$ vector defined by:

$$u_i^{(k)} = \begin{cases} 0 & \text{for } i < k \\ \sigma_k + x_{kk}^{(k)} & \text{for } i = k \\ x_{ik}^{(k)} & \text{for } i > k. \end{cases} \quad (37)$$

The matrices $Q^{(k)}$ *need not be computed explicitly because* from Equations 33 and 34:

$$X^{(k+1)} = X^{(k)} - \alpha_k U^{(k)\prime} X^{(k)} U^{(k)}. \quad (38)$$

Thus the vector $U^{(k)}$ and the scalar α_k contain all information about the orthogonal transformation $Q^{(k)}$ at step k. They may be saved for the transformation of the vector Y and for later use with iterative improvements. Additional space is needed for only the diagonal elements $u_k^{(k)}$ and for the scalars α_k; $k = 1, 2, \ldots, p$. In the method presented by Golub the elements of $U^{(k)}$ below the diagonal element $u_k^{(k)}$ and the upper triangular matrix $\tilde{R}$ are packed in the same space originally occupied by the coefficient matrix X.

Simultaneous Adjustment of Bundles Using Householder Transformations and Linked Memory Allocation

A computer program was developed for the simultaneous adjustment of bundles using Golub's scheme for the least-squares solution in conjunction with a compact storage scheme known as *linked memory allocation*. In this compact scheme, space is reserved for only the non-zero entries of the coefficient matrix without paying undue attention to their locations. Each non-zero entry of the matrix is defined by five parameters: value, row, column, address of succeeding non-zero entry in the row, and address of succeeding non-zero entry in the column.

It was necessary, however, to make a modification in Golub's scheme to economize on storage requirements. The vectors $U^{(k)}$ which are used to apply the orthogonal transformation at step k are not saved beyond that step. So these vectors are no longer available to transform the random vector Y. For the computation of the unknown parameters $\hat{\beta}$, the equation:

$$R'R\,\beta = X'Y \quad (39)$$

is used instead of Equation 27. This resulted in some additional arithmetic operations to form the column vector $X'Y$, but it relieved the memory of a substantial storage space.

Another storage problem arose during implementing Golub's scheme in conjunction with sparse matrices. Many non-zero entries were created during the intermediate steps of matrix decomposition. This outbalanced the expected benefits from the use of linked memory allocation as a compact storage scheme and imposed a severe limitation on the size of blocks that could be handled in the main memory. Further studies, however, showed that a proper preordering of the unknown parameters can substantially reduce the number of these newly created entries. A block of aerial photographs is partitioned into subblocks or strips with a minimum correlation between them. The unknown parameters, such as camera orientation elements and pass point coordinates, corresponding to each subblock are treated as one subset of unknown parameters. These subsets are then ordered to correspond to the sequence of the subblocks or strips within the block.

For example, the subblock shown in Figure 1 may be partitioned into 2 strips and the ordered set of unknown parameters is the union of 2 subsets of unknowns. The first subset consists of the orientation elements of photographs 1, 2, 3 and the coordinates of pass points 12, 21, 22, 23, 31, 32, 33. (The numbering of pass points is such that the first digit gives the row position and the second digit gives the column position.) The second subset of unknowns consists of the orientation elements of photographs 4, 5, 6 and the coordinates of pass points 41, 42, 43, 52. The correlation between the 2 strips is due to the fact that the pass points 31, 32, 33 are common. It is clear now that the ordering of unknowns given in Figure 2 is not the ideal one.

Conclusions

The following conclusions have been made from this investigation:

- The use of orthogonal transformations in the numerical solution of least squares is quite suitable for the problem of aerial triangulation adjustment. Preliminary studies indicate that the method of orthogonal transformations is numerically more stable than other direct methods of solutions. Further tests with larger blocks are necessary to verify the indication.
- The storage requirements of Householder transformations are more favorable than those of Gram-Schmidt process. Larger blocks could be simultaneously adjusted within the main memory of an electronic digital computer if Householder transformations are applied. The use of the compact storage scheme known as *linked memory allocation* proved to be useful in this respect.
- For very large blocks which cannot be handled all at once in the main memory, the use of auxiliary storage devices, which are rather slow in the read and write operations, would be inevitable. The Gram-Schmidt process would be more suitable in this instance. It would allow the reduction of very large systems of equations to smaller systems through the repeated application of the principle of partitioning a vector space into a subspace and its orthogonal complement.
- The Gram-Schmidt process offers the possibility of estimating the accuracy of the unknown parameters and computation of their covariance matrix without any matrix inversion as is usually done in the classical least squares via normal equations.

Acknowledgments

The author gratefully acknowledges the guidance of Prof. Arthur J. McNair, Cornell University, in this investigation.

Thanks are due Prof. James M. Anderson and Mr. Everett Ramey, Co-chairman of the working group of experimental research on block triangulation, Commission III of the ISP, 1968-72 for providing a set of ISP simulated data for testing the program.

The author is indebted to Dr. Richard L. Ealum for the help given for the implementation and testing of the computer program. He was himself a student at Cornell at the time this study was first undertaken.

Bibliography

1. Anderson, J. M.; McNair, A. J.; and R. L. Ealum, 1965. Analytic Aerotriangulation: Triplets and Subblocks including Use of Auxiliary Data. Phase III, Final Technical Report, Contract No. DA-44-099-AMC-502(X). GIMRADA, Cornell University, Ithaca, N.Y.
2. Bjerhammer, A., 1951. "Rectangular Reciprocal Matrices With Special Reference to Geodetic Calculations," *Bulletin Geodesique*, 52, 188-220.
3. Brown, Duane, 1968. "A Unified Lunar Control Network," *Photogrammetric Engineering*, 34:12, 1272-1292, Dec. 1968.
4. Businger, P. and G. H. Golub, 1965. "Linear Least Square Solutions by Householder Transformation." *Numerische Mathematik 7:* 269-276.
5. Forsythe, G. E. and C. B. Moler, 1967. *Computer Solution of Linear Algebraic Systems*. Prentice-Hall, Englewood Cliffs, New Jersey.
6. Householder, A. S., 1958. "Unitary Triangularization of a Nonsymmetric Matrix." *Journal Assoc. Comp. Mach.* 5: 339-342.
7. Kubik, K., 1967. *Survey of Methods in Analytical Block Triangulation*. ITC Publication A39, Netherlands.
8. Lipschutz, S., 1968. *Linear Algebra*. Schaum's Outline Series, McGraw-Hill Book Company.
9. Schmid, Hellmut, 1959. *A General Analytical Solution to the Problem of Photogrammetry*. Ballistic Research Laboratories Report No. 1065, Aberdeen Proving Ground, Md.
10. Yassa, Guirguis F.; and Arthur J. McNair, 1973. "Aerial Triangulation Viewed as a Problem of Statistical Estimation." *Proceedings*, ASP Annual Convention, Washington, D.C. March 11-16, 1973: 407-422.
11. Yassa, Guirguis F., 1973. "On the Use of Orthogonal Transformations in the Adjustment of Aerial Triangulation." *Proceedings*, ASP Annual Convention, Washington, D.C., 1973: 423-447.
12. Yassa, Guirguis F., 1974. "Orthogonal Subspaces in Analytical Triangulation." St. Louis, 1974. ACSM-ASP Convention.