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ON THE DIVISIBLE PART OF THE BRAUER GROUP OF A FIELD

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ABSTRACT. For a field k and an odd prime $p \neq \operatorname{char}(k)$ such that the p-primary component $B(k)_{(p)}$ of the Brauer group $B(k)$ of k is not zero there exists a finite extension $\tilde{k}/k$ such that $B(\tilde{k})_{(p)}$ contains a nontrivial divisible subgroup.

Let k be an arbitrary field, $p \neq \operatorname{char}(k)$ a prime, and $B(k)_{(p)}$ the p-primary component of the Brauer group $B(k)$ of k. Brumer and Rosen **[1]** conjecture that either $2B(k)_{(p)} = 0$ or $B(k)_{(p)}$ contains a nontrivial divisible subgroup. As an easy consequence of our investigation of the relative Brauer group of a maximal p-extension **[3]**, we are able to show the conjecture is true modulo a finite extension of k. For the facts about profinite groups used here, we refer the reader to **[2]**.

THEOREM. *Let k be a field and $p \neq \operatorname{char}(k)$ a prime. If $2B(k)_{(p)}$ is not zero, then there exists a finite separable extension $\tilde{k}/k$ such that $B(\tilde{k})_{(p)}$ contains a nontrivial divisible subgroup and the maximal power of p dividing $[\tilde{k} : k]$ is at most* 2.

We need a simple lemma for the proof. For V a profinite group let $V_{[p]}$ denote the smallest normal subgroup such that $V/V_{[p]}$ is a pro-p-group. Then $V_{[p]} < W_{[p]}$ for $V \leqslant W$.

LEMMA. *Let G be a profinite group, S a pro-p-subgroup, and $\mathcal{V}$ the set of open subgroups of G containing S. Then $S = \varprojlim_{V \in \mathcal{V}} V/V_{[p]}$.*

PROOF. Consider the exact sequence $1 \to V_{[p]} \to V \to V/V_{[p]} \to 1$ for $V \in \mathcal{V}$. Since $\bigcap_{V \in \mathcal{V}} V = S$ and the inverse limit is exact for profinite groups, we have only to show that $\bigcap_{V \in \mathcal{V}} V_{[p]} = 1$. Let U be an open normal subgroup of G. Then $V = SU$ is in $\mathcal{V}$ and $V_{[p]} \leqslant U$ because V/U is pro-p. So $\bigcap_{V \in \mathcal{V}} V_{[p]}$ is contained in the intersection of all open normal subgroups of G which is trivial.

PROOF OF THE THEOREM. Let k_s be the separable closure of k, with Galois group G over k, μ_{p^n} the group of p^nth roots of unity in k_s, and $\mu = \bigcup_{n=1,2,\ldots} \mu_{p^n}$. Denote by k_0 the field $k(\mu_p)$ if $p \neq 2$, and $k(\mu_4)$ if $p = 2$. Since its degree $[k_0 : k]$ divides $p - 1$ or 2, respectively, we have $B(k_0)_{(p)} \neq 0$. Hence it suffices to consider the situation $k = k_0$ and $B(k)_{(p)} \neq 0$ and to show there exists a finite extension $\tilde{k}$ of k in k_s, of degree $[\tilde{k} : k]$ prime to p, such that $B(\tilde{k})_{(p)}$ contains a nontrivial divisible subgroup.

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By [**1**, Lemma 2] we have $B(k)_{(p)} = H^2(G, \mu)$. Let S be a Sylow p-subgroup of G. Since $cd_p(G) = cd(S)$, S cannot be a free pro-p-group and hence, by the lemma, there exists an open subgroup V of G containing S, such that the group $V/V_{[p]}$ is not pro-p-free. Let $\tilde{k}$ and K be the subfields of k_s left fixed by the groups V and $V_{[p]}$, respectively. Then $[\tilde{k} : k] = [G : V]$ is prime to p and $\tilde{k}$ contains the pth or fourth roots of unity. The field K is the maximal p-extension of $\tilde{k}$ and the Galois group $\mathrm{Gal}(K/\tilde{k})$ is $V/V_{[p]}$, which is not pro-p-free. Hence, by [**3**, Satz 3(a)], the part $B(K/\tilde{k})$ of the Brauer group $B(\tilde{k})$ split by K contains a nontrivial divisible (p-primary) subgroup.

References

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