II.1 Elementary Properties and Examples

• The proof of the next proposition is similar to the proofs of the corresponding results for linear functionals in Section I.3.

Prop Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let $A: \mathcal{H} \to \mathcal{K}$ be a linear transformation. Then the following are equivalent

1. A is uniformly continuous
2. A is continuous at 0.
3. A is continuous at some point
4. There is a constant $c > 0$ such that $\frac{\|Ah\|}{\|h\|} < c$ for all $h \in \mathcal{H} \setminus \{0\}$.

Moreover, one has

$$\sup_{h \in \mathcal{H} \setminus \{0\}} \frac{\|Ah\|}{\|h\|} = \sup_{\|h\| \le 1} \|Ah\| = \sup_{\|h\|=1} \|Ah\| = \inf\{c > 0 : \|Ah\| \le c\|h\| \ \forall h \in \mathcal{H}\}.$$

Def A linear transformation $A: \mathcal{H} \to \mathcal{K}$ between two Hilbert spaces is bounded if it satisfies any (hence all) of the conditions in the above proposition. The (operator) norm of such a linear transformation is the quantity

$$\|A\| := \sup_{h \in \mathcal{H} \setminus \{0\}} \frac{\|Ah\|}{\|h\|}$$

The set of all bounded linear transformations from $\mathcal{H}$ to $\mathcal{K}$ is denoted $\mathcal{B}(\mathcal{H}, \mathcal{K})$. When $\mathcal{H} = \mathcal{K}$ we write $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H}, \mathcal{H})$.

• Observe that $\mathcal{B}(\mathcal{H}, \mathbb{F})$ is the set of all bounded linear functionals on $\mathcal{H}$.
• Also $\|Ah\| \le \|A\| \, \|h\|$ holds for all $h \in \mathcal{H}$.

Prop Let $\mathcal{H}, \mathcal{K}$, and $\mathcal{L}$ be Hilbert spaces.

1. For $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $A + B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ with $\|A + B\| \le \|A\| + \|B\|$.
2. For $\alpha \in \mathbb{F}$ and $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $\alpha A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ with $\|\alpha A\| = |\alpha| \, \|A\|$.
3. For $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $B \in \mathcal{B}(\mathcal{K}, \mathcal{L})$, $BA \in \mathcal{B}(\mathcal{H}, \mathcal{L})$ with $\|BA\| \le \|B\| \, \|A\|$.

$\mathcal{B}(\mathcal{H}, \mathcal{K})$ is a vector space

$\mathcal{B}(\mathcal{H})$ is a ring

Proof (1): Let $h \in \mathcal{H}$ with $\|h\| = 1$. Then the triangle inequality implies

$$\|(A+B)h\| = \|Ah + Bh\| \le \|Ah\| + \|Bh\| \le \|A\| + \|B\|$$

Thus $\|A+B\| \le \|A\| + \|B\|$.

(2): For any $h \in \mathcal{H}$ we have $\|(\alpha A)h\| = \|\alpha(Ah)\| = |\alpha| \, \|Ah\|$. Thus

$$\|\alpha A\| = \sup_{\|h\|=1} \|(\alpha A)h\| = |\alpha| \sup_{\|h\|=1} \|Ah\| = |\alpha| \, \|A\|.$$

(3): Let $h \in \mathcal{H}$ with $\|h\| = 1$. Then

$$\|(BA)h\| = \|B(Ah)\| \le \|B\| \, \|Ah\| \le \|B\| \, \|A\| \qquad \square$$

Ex ① Let $\mathcal{K} \le \mathcal{H}$. Recall that for $P_{\mathcal{K}}: \mathcal{H} \to \mathcal{H}$ we showed $\|P_{\mathcal{K}} h\| \le \|h\|$ $\forall h \in \mathcal{H}$. Thus $P_{\mathcal{K}} \in \mathcal{B}(\mathcal{H})$ with $\|P_{\mathcal{K}}\| \le 1$. Moreover, since $P_{\mathcal{K}} h = h$ for $h \in \mathcal{K}$, we in fact have $\|P_{\mathcal{K}}\| = 1$.

② Let $V: \mathcal{H} \to \mathcal{K}$ be an isometry. Then $\|Vh\| = \|h\|$ for all $h \in \mathcal{H}$. Thus $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ with $\|V\| = 1$. In particular, if $U: \mathcal{H} \to \mathcal{K}$ is an isomorphism then $\|U\| = 1$.

③ If $\dim \mathcal{H} = n < \infty$, then every linear transformation $A: \mathcal{H} \to \mathcal{K}$ is bounded. (Exercise: prove this.) When $\dim \mathcal{K} = m < \infty$, it follows that there is a 1-1 correspondence

$$\mathcal{B}(\mathcal{H}, \mathcal{K}) \longleftrightarrow M_{m \times n}(\mathbb{F})$$
$$A \longmapsto (\langle A e_j, f_i \rangle)_{ij}$$

where $\{e_1, \dots, e_n\} \subset \mathcal{H}$ and $\{f_1, \dots, f_m\} \subset \mathcal{K}$ are orthonormal bases. This map respects addition and scalar multiplication, and if $\mathcal{H} = \mathcal{K}$ it respects composition too.

Claim:

$$\|A\| \le \Big(\sum_{i,j} |\langle A e_j, f_i \rangle|^2 \Big)^{1/2}$$

Indeed, let $h \in \mathcal{H}$ with $\|h\| = 1$. Then we have

$$\|Ah\|^2 = \sum_i |\langle Ah, f_i \rangle|^2 = \sum_i |\langle A(\sum_j \langle h, e_j \rangle e_j), f_i \rangle|^2$$

$$= \sum_i \Big| \sum_j \langle h, e_j \rangle \langle A e_j, f_i \rangle \Big|^2$$

$$\le \sum_i \Big(\underbrace{\sum_j |\langle h, e_j \rangle|^2}_{\|h\|^2 = 1} \Big) \Big(\sum_j |\langle A e_j, f_i \rangle|^2 \Big) = \sum_{i,j} |\langle A e_j, f_i \rangle|^2.$$

However this inequality is rarely an equality. For example, if $\mathcal{H} = \mathcal{K}$ and $A = P_{\operatorname{span}\{e_1, e_2\}}$, then

$$\sum_{i,j} |\langle A e_j, e_i \rangle|^2 = 2$$

but $\|A\|^2 = 1^2$ by the previous example. □

Thm Let (X, Ω, μ) be a σ-finite measure space. For $\phi \in L^\infty(X, \Omega, \mu)$ define $M_\phi : L^2(X, \Omega, \mu) \to L^2(X, \Omega, \mu)$ by $M_\phi f = \phi f$. Then $M_\phi \in \mathcal{B}(L^2(X, \Omega, \mu))$ with $\|M_\phi\| = \|\phi\|_\infty$.

Proof For $f \in L^2(X, \Omega, \mu)$ we have

$$\int_X |M_\phi f|^2 \, d\mu = \int_X |\phi f|^2 \, d\mu = \int_X |\phi|^2 |f|^2 \, d\mu$$
$$\le \int_X \|\phi\|_\infty^2 |f|^2 \, d\mu = \|\phi\|_\infty^2 \|f\|_2^2$$

So $\|M_\phi f\|_2 \le \|\phi\|_\infty \|f\|_2$, which implies $\|M_\phi\| \le \|\phi\|_\infty$.

Let $\varepsilon>0$, then by definition of $\|\phi\|_\infty$,
$$S=\{x\in X : |\phi(x)| \geq \|\phi\|_\infty-\varepsilon\}$$
is not measure zero. Since (X,Ω,μ) is σ-finite we can find $S_0\subset S$ s.t. $0<\mu(S_0)<\infty$.
Then $1_{S_0}\in L^2(X,\Omega,\mu)$ with $\|1_{S_0}\|_2^2=\mu(S_0)$. Now
$$\frac{\|M_\phi 1_{S_0}\|_2^2}{\|1_{S_0}\|_2^2} = \frac{1}{\mu(S_0)}\int_X |\phi 1_{S_0}|^2 d\mu = \frac{1}{\mu(S_0)}\int_{S_0}|\phi|^2 d\mu \geq \frac{1}{\mu(S_0)}\int_{S_0}(\|\phi\|_\infty-\varepsilon)^2 d\mu = (\|\phi\|_\infty-\varepsilon)^2.$$
Thus
$$\|M_\phi\| = \sup_{f\in L^2(X,\Omega,\mu)\setminus\{0\}} \frac{\|M_\phi f\|}{\|f\|} \geq \|\phi\|_\infty-\varepsilon.$$

Since $\varepsilon>0$ was arbitrary, we obtain $\|M_\phi\|=\|\phi\|_\infty$ $\square$

Rem The σ-finite hypothesis is necessary for $\|M_\phi\|\geq\|\phi\|$. Indeed, let μ be the measure on $[0,1]$ s.t.
$$\mu(S) = \begin{cases} m(S) & \text{if } 0\notin S \\ \infty & \text{otherwise}\end{cases}$$
Then for $\phi=1_{\{0\}}$ we have $\|\phi\|_\infty=1$. On the other hand, for $f\in L^2([0,1],\mu)$ we have
$$\infty > \|f\|_2^2 = \int_{[0,1]}|f|^2 d\mu \geq |f(0)|\mu(\{0\})$$
so we must have $f(0)=0$. Consequently $M_\phi f = 1_{\{0\}}f=0$. So
$$\|M_\phi\|=0 < 1=\|\phi\|_\infty.$$

Thm Let (X,Ω,μ) be a measure space. Suppose $K: X\times X\to\mathbb{F}$ is a $\Omega\times\Omega$-measurable function for which there are $c_1,c_2>0$ such that
$$\int_X |k(x,y)|\,d\mu(y) \leq c_1 \text{ for } \mu\text{-a.e. } x\in X$$
and
$$\int_X |k(x,y)|\,d\mu(x) \leq c_2 \text{ for } \mu\text{-a.e. } y\in X$$
Then
$$L^2(X,\Omega,\mu)\ni f\longmapsto \int k(x,y)f(y)\,d\mu(y)$$
defines a bounded linear operator $K\in B(L^2(X,\Omega,\mu))$ with $\|K\|\leq(c_1c_2)^{1/2}$.

Proof For $f\in L^2(X,\Omega,\mu)$ we have for μ-a.e. x
$$\begin{aligned}|(Kf)(x)| &\leq \int_X |k(x,y)|\,|f(y)|\,d\mu(y)\\ &= \int_X |k(x,y)|^{1/2}\left(|k(x,y)|^{1/2}|f(y)|\right)d\mu(y)\\ &\leq \left(\int_X |k(x,y)|\,d\mu(y)\right)^{1/2}\left(\int_X |k(x,y)|\,|f(y)|^2\,d\mu(y)\right)^{1/2}\\ &\leq c_1^{1/2}\left(\int_X |k(x,y)|\,|f(y)|^2\,d\mu(y)\right)^{1/2}\end{aligned}$$

Thus
$$\int |(Kf)(x)|^2 d\mu(x) \leq c_1 \int_X\int_X |k(x,y)|\,|f(y)|^2 d\mu(y)\, d\mu(x)$$
$$= c_1 \int_X |f(y)|^2 \int_X |k(x,y)|\, d\mu(x)\, d\mu(y)$$
$$\leq c_1 c_2 \|f\|_2$$
Therefore $Kf \in L^2(X,\Omega,\mu)$ and $\|K\| \leq (c_1c_2)^{1/2}$. $\square$

Def An operator $K \in \mathcal{B}(L^2(X,\Omega,\mu))$ of the form
$$(Kf)(x) = \int_X k(x,y) f(y)\, d\mu(y) \qquad f \in L^2(X,\Omega,\mu),\ x \in X$$
is called an integral operator. The function k is called the kernel of K.

- The previous theorem is the first of several conditions on a kernel k that guarantee K is bounded.

Ex (Volterra Operator)
Consider $k: [0,1]\times[0,1] \to \mathbb{C}$, $k = 1_{\{(x,y):\, y<x\}}$. Then
$$\int_0^1 |k(x,y)|\, dm(y) = \int_0^x 1\, dy = x \leq 1$$
$$\int_0^1 |k(x,y)|\, dm(x) = \int_y^1 1\, dx = 1-y \leq 1$$
The corresponding integral operator V is called the Volterra operator:
$$(Vf)(x) = \int_0^1 k(x,y) f(y)\, dy = \int_0^x f(y)\, dy.$$

For $f \in C[0,1]$, Vf is an anti-derivative. $\square$

II.2 The Adjoint of an Operator

Def If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, a function $u: \mathcal{H}\times\mathcal{K} \to \mathbb{F}$ is called a sesquilinear form if

1. $u(\alpha h + \beta g, k) = \alpha u(h,k) + \beta u(g,k)$ for all $\alpha,\beta\in\mathbb{F}$, $h,g\in\mathcal{H}$, and $k\in\mathcal{K}$
2. $u(h, \alpha k + \beta g) = \bar{\alpha} u(h,k) + \bar{\beta} u(h,f)$ for all $\alpha,\beta\in\mathbb{F}$, $h\in\mathcal{H}$, and $g,f\in\mathcal{K}$

We say u is bounded if there exists $c\geq 0$ such that
$$|u(h,k)| \leq c\,\|h\|\,\|k\| \qquad \forall\, h\in\mathcal{H},\ k\in\mathcal{K}.$$

Ex Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. For $A\in\mathcal{B}(\mathcal{H},\mathcal{K})$ and $B\in\mathcal{B}(\mathcal{K},\mathcal{H})$
$$(h,k)\mapsto \langle Ah, k\rangle$$
$$(h,k)\mapsto \langle h, Bk\rangle$$
define bounded sesquilinear forms with bounds $\|A\|$ and $\|B\|$, respectively. $\square$

- It turns out all bounded sesquilinear forms have this form:

Thm If $u: \mathcal{H}\times\mathcal{K}\to\mathbb{F}$ is a bounded sesquilinear form with bound M, then there are unique operators $A\in\mathcal{B}(\mathcal{H},\mathcal{K})$ and $B\in\mathcal{B}(\mathcal{K},\mathcal{H})$ such that
$$u(h,k) = \langle Ah, k\rangle = \langle h, Bk\rangle \qquad \forall h\in\mathcal{H}, k\in\mathcal{K}$$
and $\|A\|, \|B\| \leq M$.

Proof Fix $k\in\mathcal{K}$ and define $L_k: \mathcal{H}\to\mathbb{F}$ by
$$L_k(h) := \overline{u(h,k)}$$
Since u is bounded so is L_k and so by the Riesz rep. theorem there exists $g\in\mathcal{H}$ such that $L_k(h) = \langle h, g\rangle$. Define $B:\mathcal{K}\to\mathcal{H}$ by $Bk = g$. Then B is linear (Exercise: verify!) and
$$\|Bk\|^2 = |\langle g,g\rangle| = |L_k(g)| = |u(g,k)| \leq M\,\|g\|\cdot\|k\| = M\,\|Bk\|\cdot\|k\|.$$
Dividing by $\|Bk\|$ yields $\|Bk\|\leq M\|k\|$ so that B is bounded with $\|B\|\leq M$.
To produce A, for $h\in\mathcal{H}$ define $L_h:\mathcal{K}\to\mathbb{F}$ by
$$L_h(k) = \overline{u(h,k)},$$
and proceed as above to obtain $A:\mathcal{H}\to\mathcal{K}$ s.t.
$$\langle Ah, k\rangle = \overline{\langle k, Ah\rangle} = \overline{L_h(k)} = u(h,k)$$
As above, we obtain $\|A\|\leq M$.
Finally, to see uniqueness suppose $A_1\in\mathcal{B}(\mathcal{H},\mathcal{K})$ satisfies $\langle A_1 h, k\rangle = u(h,k)$ for all $h\in\mathcal{H}$ and $k\in\mathcal{K}$. Then $\langle (A-A_1)h, k\rangle = 0$ for all $h\in\mathcal{H}$, $k\in\mathcal{K}$. This implies $(A-A_1)h = 0$ for all $h\in\mathcal{H}$ and so $A = A_1$. A similar argument shows B is unique as well. $\square$

Def For $A\in\mathcal{B}(\mathcal{H},\mathcal{K})$, the unique $B\in\mathcal{B}(\mathcal{K},\mathcal{H})$ satisfying

$$\langle Ah, k\rangle = \langle h, Bk\rangle$$

for all $h\in\mathcal{H}$ and $k\in\mathcal{K}$ is called the adjoint of A and is denoted $A^* := B$.

Ex For $A\in M_{m\times n}(\mathbb{C})$, thought of as $A\in\mathcal{B}(\mathbb{C}^n,\mathbb{C}^m)$, A^* is precisely the conjugate transpose. For $A\in M_{m\times n}(\mathbb{R})$, $A^* = A^T$. $\square$

Prop Let $U:\mathcal{H}\to\mathcal{K}$ be an isomorphism. Then $U^* = U^{-1}$.

Proof We simply observe

$$\langle Uh, k\rangle = \langle Uh, U(U^{-1}k)\rangle = \langle h, U^{-1}k\rangle$$

for $h\in\mathcal{H}$ and $k\in\mathcal{K}$. The uniqueness of U^* implies $U^* = U^{-1}$. $\square$

- We will now now focus on $\mathcal{B}(\mathcal{H})$, which has the advantage of being an algebra (ring).

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Prop For $A, B\in\mathcal{B}(\mathcal{H})$ and $\alpha\in\mathbb{F}$

1. $(\alpha A + B)^* = \bar{\alpha} A^* + B^*$
2. $(AB)^* = B^*A^*$
3. $(A^*)^* = A$
4. If A has a bounded inverse $A^{-1}\in\mathcal{B}(\mathcal{H})$, then A^* has bounded inverse $(A^{-1})^*$.
5. $\|A\| = \|A^*\| = \|A^*A\|^{1/2} = \|AA^*\|^{1/2}$. $\square$

Proof (1) – (4) are left as exercises.

(5) For $h\in\mathcal{H}$ we have

$$\|Ah\|^2 = \langle Ah, Ah\rangle = \langle A^*Ah, h\rangle \le \|A^*Ah\|\,\|h\| \le \|A^*A\|\,\|h\|^2 \le \|A^*\|\,\|A\|\,\|h\|^2.$$

Thus

$$(*)\qquad \|A\|^2 \le \|A^*A\| \le \|A^*\|\,\|A\|.$$

In particular, dividing by $\|A\|$ yields $\|A\|\le\|A^*\|$. Repeating this argument with A and A^* reversed yields

$$(**)\qquad \|A^*\|^2 \le \|AA^*\| \le \|A\|\,\|A^*\|$$

and in particular $\|A^*\|\le\|A\|$. Thus $\|A\| = \|A^*\|$ and plugging this into $(*)$ and $(**)$ yields the other equalities. $\square$

Rem A deep result called the open mapping theorem, which we will prove in Chapter III, will imply the "bounded" hypothesis in (4) is superfluous. That is, so long as $A\in\mathcal{B}(\mathcal{H})$ is a bijection, its inverse will automatically be bounded.
The equalities in (5) are called the C^*-identity and are an axiom of the theory of C^*-algebras.

Ex (1) For $\phi\in L^\infty(X,\Omega,\mu)$, $M_\phi^* = M_{\bar{\phi}}$.

(2) For integral operator K with kernel k, K^* is the int. op with kernel $k^*(x,y) := \overline{k(y,x)}$.

(3) Let $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the isometry

$$S(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots)$$

As an isometry $S \in \mathcal{B}(\ell^2(\mathbb{N}))$ with $\|S\| = 1$. Also we have

$$\langle S(x_1, x_2, \dots), (y_1, y_2, \dots) \rangle = \langle (0, x_1, x_2, \dots), (y_1, y_2, y_3, \dots) \rangle$$
$$= \langle (x_1, x_2, \dots), (y_2, y_3, \dots) \rangle$$

Thus $S^*(y_1, y_2, y_3, \dots) = (y_2, y_3, \dots)$. S is called the unilateral shift and S^* is called the backward shift. □

Def Let $A \in \mathcal{B}(\mathcal{H})$. We say A is

(1) self-adjoint (or hermitian) if $A^* = A$;

(2) unitary if $A^*A = AA^* = 1$ (i.e. $A^* = A^{-1}$)

(3) normal if $A^*A = AA^*$.

- Note that self-adjoint and unitary operators are normal.

- One shall think of taking the adjoint as the analogue of taking the complex conjugate. Indeed, when $\mathcal{H} = \mathbb{C}$ $\mathcal{B}(\mathbb{C}) = \{\mathbb{C} \ni z \mapsto \alpha z : \alpha \in \mathbb{C}\}$ and $(z \mapsto \alpha z)^* = (z \mapsto \bar{\alpha} z)$. With this point of view, self-adjoint operators play the role of $\mathbb{R}$ and unitary operators play the role of $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Ex (1) M_ϕ is always normal since $\phi\bar{\phi} = \bar{\phi}\phi = |\phi|^2$. It is self-adjoint iff $\phi = \bar{\phi}$, i.e. ϕ is $\mathbb{R}$-valued. It is unitary iff $|\phi|^2 = 1$, i.e. ϕ is $\mathbb{T}$ valued.

(2) An integral operator with kernel k is self-adjoint iff $k(x,y) = \overline{k(y,x)}$ for $\mu \times \mu$ - a.e. (x,y).

(3) Let S be the unilateral shift then $S^*S = 1$, while

$$SS^*(x_1, x_2, x_3, \dots) = (0, x_2, x_3, \dots)$$

Thus S is not normal. □

Prop If $\mathcal{H}$ is a Hilbert space over $\mathbb{C}$, then $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint iff $\langle Ah, h \rangle \in \mathbb{R}$ for all $h \in \mathcal{H}$.

Proof ($\Rightarrow$) We have

$$\overline{\langle Ah, h \rangle} = \langle h, Ah \rangle = \langle A^*h, h \rangle = \langle Ah, h \rangle$$

Thus $\overline{\langle Ah, h \rangle} = \langle Ah, h \rangle$ which implies $\langle Ah, h \rangle \in \mathbb{R}$.

($\Leftarrow$) For $\alpha \in \mathbb{C}$ and $h, g \in \mathcal{H}$ we have

$$\mathbb{R} \ni \langle A(h + \alpha g), (h + \alpha g) \rangle = \underbrace{\langle Ah, h \rangle}_{\in \mathbb{R}} + \alpha \langle Ag, h \rangle + \bar{\alpha} \langle Ah, g \rangle + \underbrace{|\alpha|^2 \langle Ag, g \rangle}_{\in \mathbb{R}}$$

Thus $\alpha \langle Ag, h \rangle + \bar{\alpha} \langle Ah, g \rangle \in \mathbb{R}$. Taking the complex conjugate yields

$$\alpha\langle Ag,h\rangle+\bar{\alpha}\langle Ah,g\rangle = \bar{\alpha}\langle h,Ag\rangle+\alpha\langle g,Ah\rangle = \bar{\alpha}\langle A^*h,g\rangle+\alpha\langle A^*g,h\rangle$$

Substituting $\alpha=1$ and $\alpha=i$ yields

$$\langle Ag,h\rangle+\langle Ah,g\rangle = \langle A^*h,g\rangle+\langle A^*g,h\rangle$$
$$i\langle Ag,h\rangle-i\langle Ah,g\rangle = -i\langle A^*h,g\rangle+i\langle A^*g,h\rangle$$

Multiplying the second equation by $-i$ and adding it to the first yields $2\langle Ag,h\rangle=2\langle A^*g,h\rangle$. It follows that $A=A^*$. $\square$

Rem The above proposition doesn't hold for Hilbert spaces over $\mathbb{R}$. Indeed, for $A\in M_n(\mathbb{R})$, $\langle Ah,g\rangle\in\mathbb{R}$ for all $h,g\in\mathbb{R}^n$. So in particular, $\langle Ah,h\rangle\in\mathbb{R}$. However $A^*=A^T$, so A is self-adjoint iff $A=A^T$.

Prop For $A=A^*\in B(\mathcal{H})$

$$\|A\| = \sup_{\|h\|=1}|\langle Ah,h\rangle|$$

Proof Denote the right-side above by M. Then the Cauchy-Schwarz inequality says

$$|\langle Ah,h\rangle| \le \|Ah\|\cdot\|h\| \le \|A\|\cdot\|h\|\cdot\|h\| = \|A\|.$$

So $M\le\|A\|$.

Conversely, observe for $h,g\in\mathcal{H}$ with $\|h\|=1=\|g\|$

$$\langle A(h\pm g),h\pm g\rangle = \langle Ah,h\rangle\pm\langle Ah,g\rangle\pm\langle Ag,h\rangle+\langle Ag,g\rangle$$
$$= \langle Ah,h\rangle\pm\langle Ah,g\rangle\pm\langle g,Ah\rangle+\langle Ag,g\rangle$$
$$= \langle Ah,h\rangle\pm 2\,\mathrm{Re}\langle Ah,g\rangle+\langle Ag,g\rangle$$

Consequently

$$\mathrm{Re}\langle Ah,g\rangle=\tfrac{1}{4}(\langle A(h+g),h+g\rangle-\langle A(h-g),h-g\rangle)$$

Now, $|\langle Af,f\rangle|\le M\|f\|^2$ for all $f\in\mathcal{H}$ (simply normalize f if non-zero). Thus the above equation and the Parallelogram Law imply

$$\mathrm{Re}\langle Ah,g\rangle \le \tfrac{1}{4}M(\|h+g\|^2+\|h-g\|^2)$$
$$= \tfrac{1}{4}M\,2(\|h\|^2+\|g\|^2) = M.$$

In particular, for $g=\frac{1}{\|Ah\|}Ah$ we obtain

$$M \ge \mathrm{Re}\langle Ah,\tfrac{1}{\|Ah\|}Ah\rangle = \tfrac{1}{\|Ah\|}\mathrm{Re}\langle Ah,Ah\rangle = \|Ah\|.$$

Thus $\|A\|\le M$ and so $\|A\|=M$. $\square$

Cor If $A=A^*\in B(\mathcal{H})$ and $\langle Ah,h\rangle=0$ for all $h\in\mathcal{H}$, then $A=0$.

• Note that if $\mathcal{H}$ is over $\mathbb{C}$, then $\langle Ah,h\rangle=0\in\mathbb{R}$ for all $h\in\mathcal{H}$ implies $A=A^*$. However, if $\mathcal{H}$ is over $\mathbb{R}$ then $A=A^*$ is a necessary hypothesis:

Ex Let S be the unilateral shift on $\ell^2(\mathbb{N},\mathbb{R})$. Then for $A=S-S^*$ we have $A\ne0$ and

$$\langle A(x_1,x_2,\dots),(x_1,x_2,\dots)\rangle = \langle (-x_2,x_1-x_3,\dots),(x_1,x_2,\dots)\rangle$$
$$= -x_2x_1 + x_1x_2 - x_3x_2 + \cdots \quad = 0$$

This fails if $(x_n)\in \ell^2(\mathbb{N},\mathbb{C})$. $\square$

- Given $A\in B(\mathcal{H})$ with $\mathcal{H}$ a Hilbert space over $\mathbb{C}$ we have that
$$\tfrac{1}{2}(A+A^*) \quad \text{and} \quad \tfrac{1}{2i}(A-A^*)$$
are self-adjoint with
$$A = \tfrac{1}{2}(A+A^*) + i\,\tfrac{1}{2i}(A-A^*)$$
We call $\operatorname{Re}A := \frac{1}{2}(A+A^*)$ the real part of A and $\operatorname{Im}A := \frac{1}{2i}(A-A^*)$ the imaginary part.
Note A is self-adjoint iff $\operatorname{Re}A = A$ iff $\operatorname{Im}A = 0$.

Prop For $A\in B(\mathcal{H})$, the following are equivalent:
1. A is normal.
2. $\|Ah\| = \|A^*h\|$ for all $h\in\mathcal{H}$.

If $\mathcal{H}$ is over $\mathbb{C}$, then these are further equivalent to
3. The real and imaginary parts of A commute.

Proof $(1)\iff(2)$: Observe that $A^*A - AA^*$ is self-adjoint (for any A). Also
$$\langle (A^*A-AA^*)h,h\rangle = \langle A^*Ah,h\rangle - \langle AA^*h,h\rangle = \|Ah\|^2 - \|A^*h\|^2$$
Now A is normal $\iff A^*A-AA^*=0$ and by the previous corollary that is equivalent to the above being zero for all $h\in\mathcal{H}$.
$(1)\iff(3)$: Observe that
$$A^*A = (\operatorname{Re}A - i\operatorname{Im}A)(\operatorname{Re}A + i\operatorname{Im}A)$$
$$= (\operatorname{Re}A)^2 + i\operatorname{Re}A\operatorname{Im}A - i\operatorname{Im}A\operatorname{Re}A + (\operatorname{Im}A)^2$$
$$AA^* = (\operatorname{Re}A + i\operatorname{Im}A)(\operatorname{Re}A - i\operatorname{Im}A)$$
$$= (\operatorname{Re}A)^2 - i\operatorname{Re}A\operatorname{Im}A + i\operatorname{Im}A\operatorname{Re}A + (\operatorname{Im}A)^2$$
Thus
$$A^*A - AA^* = 2i\operatorname{Re}A\operatorname{Im}A - 2i\operatorname{Im}A\operatorname{Re}A$$
Consequently A is normal iff $\operatorname{Re}A\operatorname{Im}A = \operatorname{Im}A\operatorname{Re}A$. $\square$

Prop For $A\in B(\mathcal{H})$, the following are equivalent
1. A is an isometry
2. $A^*A = 1$

Proof Recall that we previously showed A is an isometry iff $\langle Ah,Ag\rangle = \langle h,g\rangle$ for all $h,g\in\mathcal{H}$. Then the equivalence of (1) and (2) follows from
$$\langle (A^*A-1)h,g\rangle = \langle Ah,Ag\rangle - \langle h,g\rangle$$
for all $h,g\in\mathcal{H}$. $\square$

Prop For $A \in B(\mathcal{H})$, the following are equivalent

1. A is unitary.
2. A is an isomorphism.
3. A is a normal isometry

Proof $(1) \Rightarrow (2)$: The previous proposition implies A is an isometry. Since $A^* = A^{-1}$, it follows that is also surjective. We previously noted surjective isometries are isomorphisms.

$(2) \Rightarrow (3)$ Since A is an isometry, $A^*A = I$ by the previous proposition. But then

$$A^{-1} = (A^*A)A^{-1} = A^*(AA^{-1}) = A^*$$

so $A^{-1} = A^*$ and consequently $AA^* = AA^{-1} = 1 = A^*A$. That is, A is normal.

$(3) \Rightarrow (1)$ By the previous proposition, $A^*A = 1$. Since A is normal we also have $AA^* = 1$. So A is unitary. $\square$

Thm For $A \in B(\mathcal{H})$,

$$\ker A = (\operatorname{ran} A^*)^\perp \qquad \ker A^* = (\operatorname{ran} A)^\perp \qquad \overline{\operatorname{ran} A} = (\ker A^*)^\perp \qquad \overline{\operatorname{ran} A^*} = (\ker A)^\perp$$

Proof Let $h \in \ker A$, then for any $g \in \mathcal{H}$

$$\langle h, A^*g \rangle = \langle Ah, g \rangle = 0$$

Thus $\ker A \subset (\operatorname{ran} A^*)^\perp$. Conversely, if $h \in (\operatorname{ran} A^*)^\perp$ then

$$\langle Ah, g \rangle = \langle h, A^*g \rangle = 0$$

for all $g \in \mathcal{H}$ so that $Ah = 0$. Thus $(\operatorname{ran} A^*)^\perp = \ker A$.

The second equality holds by applying the above to A^* and using $A^{**} = A$. The remaining two equalities follow by taking orthogonal complements in the first two. $\square$

II.3 Projections, Idempotents, Invariant and Reducing Subspaces

Def We say $E \in B(\mathcal{H})$ is an *idempotent* if $E^2 = E$. A *projection* is an idempotent P such that $\ker P = (\operatorname{ran} P)^\perp$

Ex ① For any $\mathcal{K} \leq \mathcal{H}$, we've already seen $P_\mathcal{K}^2 = P_\mathcal{K}$, so it is an idempotent. Since

$$\ker P_\mathcal{K} = \mathcal{K}^\perp = (\operatorname{ran} P_\mathcal{K})^\perp$$

it is also a projection.

② $E = \begin{pmatrix} 1 & \alpha \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{F})$ is an idempotent, but for $\alpha \neq 0$ it is not a projection:
$\ker E = \{ \begin{pmatrix} -\alpha x \\ x \end{pmatrix} : x \in \mathbb{F} \}$ and $\operatorname{ran} E = \{ \begin{pmatrix} x \\ 0 \end{pmatrix} : x \in \mathbb{F} \}$. $\square$

Prop ① $E \in B(\mathcal{H})$ is an idempotent iff $1 - E$ is an idempotent.
② $\operatorname{ran} E = \ker(1-E)$ and $\ker E = \operatorname{ran}(1-E)$ and all four subspaces are closed.
③ $\operatorname{ran} E \cap \ker E = \{0\}$ and $\operatorname{ran} E + \ker E = \mathcal{H}$.

Proof (1): This follows from: $(1-E)^2 - (1-E) = 1 - 2E + E^2 - 1 + E = E^2 - E$.

(2): Since $(1-E)E = E - E^2 = 0$, $\operatorname{ran} E \subset \ker(1-E)$. Conversely, if $h \in \ker(1-E)$ then

$$h = h - Eh + Eh = (1-E)h + Eh = Eh.$$

So $\operatorname{ran} E = \ker(1-E)$. By ① we can reverse the roles of E and $1-E$ to obtain the other equality. Since E and $1-E$ are bounded (and hence continuous) they have closed kernels and therefore the ranges are closed too.

(3) For any $h \in \mathcal{H}$ we have

$$h = (1 - E + E)h = (1-E)h + Eh$$

So $\mathcal{H} = \operatorname{ran}(1-E) + \operatorname{ran} E \overset{②}{=} \ker E + \operatorname{ran} E$. In particular, if $h \in \operatorname{ran} E \cap \ker E = \ker(1-E) \cap \ker E$ we have

$$h = (1-E)h + Eh = 0$$

Thus $\operatorname{ran} E \cap \ker E = \{0\}$. $\square$

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Prop If $E \in B(\mathcal{H})$ is a non-zero idempotent, the following are equivalent:
① E is a projection
② $E = P_{\operatorname{ran} E}$.
③ $\|E\| = 1$
④ E is self-adjoint
⑤ E is normal
⑥ $\langle Eh, h \rangle \geq 0$ for all $h \in \mathcal{H}$.

Proof (1) $\Rightarrow$ (2): For $h \in \mathcal{H}$ we have $Eh \in \operatorname{ran} E$ and $h - Eh = (1-E)h \in \operatorname{ran}(1-E)$. By the previous proposition $\operatorname{ran}(1-E) = \ker E$ and by ① this is $(\operatorname{ran} E)^\perp$. Thus $E = P_{\operatorname{ran} E}$.
(2) $\Rightarrow$ (3): We have already observed this for orthogonal projections.

(3) $\Rightarrow$ (1): Let $h \in (\ker E)^\perp$. By the previous proposition, $h - Eh \in \operatorname{ran}(1-E) = \ker E$. Thus
$$0 = \langle h - Eh, h\rangle = \|h\|^2 - \langle Eh, h\rangle$$
so
$$\|h\|^2 = \langle Eh, h\rangle \leq \|Eh\|\,\|h\| \leq \|h\|^2$$
So the above are equalities:
$$\|h\| = \|Eh\| = \langle Eh, h\rangle^{1/2} \qquad h \in (\ker E)^\perp.$$
Consequently
$$\|h - Eh\|^2 = \|h\|^2 - 2\operatorname{Re}\langle Eh, h\rangle + \|Eh\|^2 = 0$$
That is $(\ker E)^\perp \subset \ker(1-E) = \operatorname{ran} E$. Conversely, for $g \in \operatorname{ran} E$ we can write $g = g_1 + g_2$ for $g_1 \in \ker E$ and $g_2 \in (\ker E)^\perp$. Then $E^2 = E$ implies
$$g = Eg = E(g_1 + g_2) = Eg_2 = g_2.$$
Thus $\operatorname{ran} E = (\ker E)^\perp$ and E is a projection.

(1) $\Rightarrow$ (4): Let $h, g \in \mathcal{H}$ and write $h = h_1 + h_2$ and $g = g_1 + g_2$ where $h_1, g_1 \in \operatorname{ran} E$ and $h_2, g_2 \in (\operatorname{ran} E)^\perp = \ker E$. Then
$$\langle Eh, g\rangle = \langle Eh_1, g\rangle = \langle Eh_1, g_1\rangle = \langle h_1, g_1\rangle$$
and
$$\langle h, Eg\rangle = \langle h, Eg_1\rangle = \langle h_1, Eg_1\rangle = \langle h_1, g_1\rangle$$
Thus $E^* = E$.

(4) $\Rightarrow$ (5): Immediate.

(5) $\Rightarrow$ (1): By a Proposition from Section II.2 we have $\|Eh\| = \|E^* h\|$ for all $h \in \mathcal{H}$. Consequently $\ker E = \ker E^* = (\operatorname{ran} E)^\perp$ by the Theorem at the end of Section II.2.

(2) $\Rightarrow$ (6) For $h \in \mathcal{H}$ write $h = h_1 + h_2$ for $h_1 \in \operatorname{ran} E$ and $h_2 \in (\operatorname{ran} E)^\perp = \ker E$. Then
$$\langle Eh, h\rangle = \langle Eh, h_1\rangle = \langle Eh_1, h_1\rangle = \|h_1\|^2 \geq 0.$$

(6) $\Rightarrow$ (1) Let $h \in \operatorname{ran} E$ and $g \in \ker E$. Then
$$0 \leq \langle E(h+g), h+g\rangle = \langle Eh, h+g\rangle = \|h\|^2 + \langle h, g\rangle$$
Suppose, towards a contradiction, that there exist $h \in \operatorname{ran} E$ and $g \in \ker E$ s.t. $\alpha := \langle h, g\rangle \neq 0$. Replacing g in the above inequality with $-\frac{2\|h\|^2}{\alpha} g$ yields
$$0 \leq \|h\|^2 - \frac{2\|h\|^2}{\alpha}\langle h, g\rangle = \|h\|^2 - 2\|h\|^2$$
a contradiction. Thus $\operatorname{ran} E \perp \ker E$ which implies $\operatorname{ran} E \subset (\ker E)^\perp$. Now if $h \in (\ker E)^\perp$, then (3) in the previous proposition implies $h = h_1 + h_2$ for $h_1 \in \operatorname{ran} E$ and $h_2 \in \ker E$. But then the above implies $h_2 = h - h_1 \in \ker E \cap (\ker E)^\perp = \{0\}$. Thus $h = h_1 \in \operatorname{ran} E$. $\square$

$\boxed{\text{Rem}}$ By the previous proposition, all projections are orthogonal projections (but of course not all idempotents are). Thus from now on we will simply say "projection" to mean "orthogonal projection."

⚠ Some people use "projection" to mean "idempotent."

Def For projections $P_1, P_2 \in B(H)$, we say P_1 is orthogonal to P_2 if $P_1P_2 = 0$.

- Note that if P_1 is orthogonal to P_2 then
$$0 = (P_1P_2)^* = P_2^*P_1^* = P_2P_1$$
by the previous proposition. So P_1 and P_2 are orthogonal to each other. We will also say P_1 and P_2 are orthogonal projections to mean $P_1P_2 = 0$, which in light of the above remark should not be confused with "P_1 and P_2 are each orthogonal projections."

Cor For projections $P_1, P_2 \in B(H)$, P_1 is orthogonal to P_2 iff $\operatorname{ran} P_1 \perp \operatorname{ran} P_2$.

Proof Let $K_i := \operatorname{ran} P_i$, so that the proposition implies $P_i = P_{K_i}$ for each $i = 1, 2$. For any $h \in H$, $P_1P_2h = 0 \iff P_2h \in K_1^\perp$. Hence $P_1P_2 = 0 \iff K_2 \subseteq K_1^\perp \iff K_1 \perp K_2$ $\square$

- Let P be a projection with $K := \operatorname{ran} P$ and $M := \ker P$. Since P is an idempotent both subspaces are closed and hence Hilbert spaces. The map
$$U : K \oplus M \longrightarrow H$$
$$(f, g) \mapsto f + g$$
is a isomorphism. Thus we can identify $H = K \oplus M$.

Notation Let $\{K_i : i \in I\}$ be a collection of closed subspaces that are pairwise orthogonal. Then we write
$$\bigoplus_{i \in I} K_i = \overline{\operatorname{span}} \bigcup_{i \in I} K_i.$$
When $|I| < \infty$,
$$\bigoplus_{i \in I} K_i = \operatorname{span} \bigcup_{i \in I} K_i = \sum_{i \in I} K_i$$
but for infinite I we must take the closure.
Also, for $K, M \leq H$ we write
$$K \ominus M := K \cap M^\perp;$$
and call this the orthogonal difference of K and M. If $M \leq K$, then $K = M \oplus (K \ominus M)$

Def For $A \in B(H)$ and $K \leq H$, we say K is an invariant subspace for A if $AK \subseteq K$. We say K is a reducing subspace for A if $AK \subseteq K$ and $AK^\perp \subseteq K^\perp$.

- Let H, K be Hilbert spaces. Then for $W \in B(H)$, $X \in B(K, H)$, $Y \in B(H, K)$, $Z \in B(K)$
$$\begin{pmatrix} W & X \\ Y & Z \end{pmatrix} \in B(H \oplus K)$$
where for $(h, k) \in H \oplus K$
$$\begin{pmatrix} W & X \\ Y & Z \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} Wh + Xz \\ Yh + Zh \end{pmatrix}$$

- Now, consider $\mathcal{K} \le \mathcal{H}$ and let $P := P_{\mathcal{K}}$. Let $U: \mathcal{K} \oplus \mathcal{K}^{\perp} \to \mathcal{H}$ be the isomorphism $U(f,g) = f+g$. Then for any $A \in \mathcal{B}(\mathcal{H})$

$$AU = U \begin{pmatrix} PAP & PA(1-P) \\ (1-P)AP & (1-P)A(1-P) \end{pmatrix}$$

Indeed, for $f \in \mathcal{K}, g \in \mathcal{K}^{\perp}$

$$U \begin{pmatrix} PAP & PA(1-P) \\ (1-P)AP & (1-P)A(1-P) \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = U \begin{pmatrix} PA(Pf + (1-P)g) \\ (1-P)A(Pf + (1-P)g) \end{pmatrix} = U \begin{pmatrix} PA(f+g) \\ (1-P)A(f+g) \end{pmatrix}$$

$$= PA(f+g) + (1-P)A(f+g) = A(f+g) = AU\begin{pmatrix} f \\ g \end{pmatrix}.$$

Thus if we identify $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^{\perp}$, then under this identification

$$A = \begin{pmatrix} PAP & PA(1-P) \\ (1-P)AP & (1-P)A(1-P) \end{pmatrix}$$

(Note also that in $\mathcal{B}(\mathcal{H})$ we have $PAP + PA(1-P) + (1-P)AP + (1-P)A(1-P) = A$.)

Prop Let $\mathcal{K} \le \mathcal{H}$ and denote $P := P_{\mathcal{K}}$. For $A \in \mathcal{B}(\mathcal{H})$, the following are equivalent:

1. $\mathcal{K}$ is invariant for A
2. $PAP = AP$
3. Under the identification $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^{\perp}$,
$$A = \begin{pmatrix} PAP & PA(1-P) \\ 0 & (1-P)A(1-P) \end{pmatrix}$$

Proof (1) $\Rightarrow$ (2): For $h \in \mathcal{H}$, $Ph \in \mathcal{K}$, so $APh = A(Ph) \in \mathcal{K}$. Consequently $P(APh) = APh$.

(2) $\Rightarrow$ (3) We have

$$(1-P)AP = AP - PAP = 0.$$

(3) $\Rightarrow$ (1) This implies $(1-P)AP = 0$. Let $k \in \mathcal{K}$, then

$$(1-P)Ak = (1-P)A(Pk) = 0$$

Thus $Ak \in (\mathcal{K}^{\perp})^{\perp} = \mathcal{K}$. So $\mathcal{K}$ is invariant for A. $\square$

Prop Let $\mathcal{K} \le \mathcal{H}$ and denote $P := P_{\mathcal{K}}$. For $A \in \mathcal{B}(\mathcal{H})$, the following are equivalent:

1. $\mathcal{K}$ is reducing for A.
2. $PA = AP$
3. Under the identification $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^{\perp}$,
$$A = \begin{pmatrix} PAP & 0 \\ 0 & (1-P)A(1-P) \end{pmatrix}$$
4. $\mathcal{K}$ is invariant for A and A^*.

Proof (1) $\Rightarrow$ (2) For $h \in \mathcal{H}$ write $h = h_1 + h_2$ where $h_1 \in \mathcal{K}$ and $h_2 \in \mathcal{K}^{\perp}$. Then $Ah_1 \in \mathcal{K}$ and $Ah_2 \in \mathcal{K}^{\perp}$ so that

$$PAh = P(Ah_1 + Ah_2) = Ah_1 = APh.$$

(2) $\Rightarrow$ (3) It suffices to show $PA(1-P) = (1-P)AP = 0$. Note

$$PA = AP \Rightarrow \begin{cases} PA = PAP \\ AP = PAP \end{cases} \Rightarrow \begin{cases} PA - PAP = 0 \\ AP - PAP = 0 \end{cases} \Rightarrow \begin{cases} PA(1-P) = 0 \\ (1-P)AP = 0 \end{cases}.$$

(3) $\Rightarrow$ (4) By the previous prop, $\mathcal{K}$ is invariant for A. Also $PA(1-P)=0$ implies $(1-P)A^*P = 0$. So the previous prop implies $\mathcal{K}$ is invariant for A^*.

(4) $\Rightarrow$ (1) It suffices to show $A\mathcal{K}^\perp \subset \mathcal{K}^\perp$. For $h \in \mathcal{K}^\perp$,

$$\langle Ah, k \rangle = \langle h, A^*k \rangle = 0 \qquad \forall k \in \mathcal{K}$$

Thus $Ah \in \mathcal{K}^\perp$. $\square$

- 3 implies that if $\mathcal{K}$ is reducing for A then A is determined by its restrictions to $\mathcal{K}$ and $\mathcal{K}^\perp$: PAP and $(1-P)A(1-P)$.

II.4 Compact Operators

- While general bounded operators on infinite dimensional Hilbert spaces can exhibit pathological behavior compared to matrices, "compact" operators behave in many ways like matrices.

Def A linear transformation $T: \mathcal{H} \to \mathcal{K}$ is compact if $\overline{T(\overline{B}(0,1))}$ is compact. The set of compact operators from $\mathcal{H}$ to $\mathcal{K}$ is denoted $K(\mathcal{H},\mathcal{K})$, and $K(\mathcal{H}) := K(\mathcal{H},\mathcal{H})$.

Lemma Let (X,d) be a complete metric space. For $S \subset X$, $\overline{S}$ is compact iff S is totally bounded.

Proof ($\Rightarrow$) $\overline{S}$ is totally bounded by virtue of being compact. Consequently S is totally bounded.

($\Leftarrow$) It suffices to show $\overline{S}$ is complete and totally bounded. Since X is complete, $\overline{S}$ is complete as a closed subspace. To see it is totally bounded, let $\varepsilon > 0$. Then there exists $x_1, \ldots, x_N \in X$ such that

$$S \subset \bigcup_{n=1}^{N} B(x_n, \varepsilon/2) \subset \bigcup_{n=1}^{N} \overline{B}(x_n, \varepsilon/2)$$

This implies

$$\overline{S} \subset \bigcup_{n=1}^{N} \overline{B}(x_n, \varepsilon/2) \subset \bigcup_{n=1}^{N} B(x_n, \varepsilon)$$

So $\overline{S}$ is totally bounded. $\square$

Prop ① $K(\mathcal{H},\mathcal{K}) \subset B(\mathcal{H},\mathcal{K})$.

② $K(\mathcal{H},\mathcal{K})$ is a vector subspace of $B(\mathcal{H},\mathcal{K})$ that is closed with respect to $\|\cdot\|$.

③ For $A \in B(\mathcal{H})$, $B \in B(\mathcal{K})$, and $T \in K(\mathcal{H},\mathcal{K})$ we have $TA, BT \in K(\mathcal{H},\mathcal{K})$

Proof (1): $T(\overline{B}(0,1))$ is bounded since its closure is compact. Thus

$$\|T\| = \sup_{\|h\| \le 1} \|Th\| < \infty$$

(2): Let $\alpha \in \mathbb{F}$ $T, S \in K(\mathcal{H},\mathcal{K})$. To see $\alpha T + S \in K(\mathcal{H},\mathcal{K})$ it suffices, by the lemma, to show $C := (\alpha T + S)(\overline{B}(0,1))$ is totally bounded. Let $\varepsilon > 0$, then the compactness of T and S imply $\exists k_1, \ldots, k_N, k_1', \ldots, k_M' \in \mathcal{K}$ s.t.

$$T(\overline{B}(0,1)) \subset \bigcup_{i=1}^{N} B\left(k_i, \tfrac{\varepsilon}{|\alpha|+1}\right)$$

$$S(\overline{B}(0,1)) \subset \bigcup_{j=1}^{M} B\left(k_j', \tfrac{\varepsilon}{|\alpha|+1}\right)$$

Suppose $T(h) \in B(k_i, \frac{\varepsilon}{|\alpha|+1})$ and $S(h) \in B(k_j', \frac{\varepsilon}{|\alpha|+1})$ for $h \in \overline{B}(0,1)$. Then

$$\|(\alpha T + S)(h) - (\alpha k_i + k_j')\| \le |\alpha| \|Th - k_i\| + \|Sh - k_j'\|$$
$$< |\alpha| \frac{\varepsilon}{|\alpha|+1} + \frac{\varepsilon}{|\alpha|+1} = \varepsilon.$$

Thus

$$(\alpha T + S)(\overline{B}(0,1)) \subset \bigcup_{\substack{1 \le i \le N \\ 1 \le j \le M}} B(\alpha k_i + k_j', \varepsilon)$$

and $(\alpha T+S)(\overline{B}(0,1))$ is totally bounded.

Now, suppose for $(T_n)_{n\in\mathbb{N}} \subset K(H,K)$ that $\exists\, T\in B(H,K)$ s.t. $\|T-T_n\|\to 0$. We must show $T\in K(H,K)$, and again it suffices to show $T(\overline{B}(0,1))$ is totally bounded. Let $\varepsilon>0$, and let $n\in\mathbb{N}$ be s.t. $\|T-T_n\|<\frac{\varepsilon}{2}$. Since T_n is compact, $\exists\, k_1,\dots,k_N\in K$ s.t.

$$T_n(\overline{B}(0,1))\subset \bigcup_{i=1}^{N} B(k_i,\tfrac{\varepsilon}{2}).$$

For $h\in\overline{B}(0,1)$, suppose $T_n h\in B(k_i,\frac{\varepsilon}{2})$. Then

$$\begin{aligned}\|Th-k_i\| &\le \|Th-T_nh\| + \|T_nh-k_i\|\\ &< \|T-T_n\|\cdot\|h\| + \tfrac{\varepsilon}{2} < \varepsilon.\end{aligned}$$

Thus

$$T(\overline{B}(0,1))\subset \bigcup_{i=1}^{N} B(k_i,\varepsilon)$$

and so $T\in K(H,K)$.

(3): Note that

$$TA(\overline{B}(0,1))\subseteq T(\overline{B}(0,\|A\|)) = \tfrac{1}{\|A\|}T(\overline{B}(0,1))$$

Since the latter is totally bounded, so is the former. Hence TA is compact.

Also note that for $k\in K$

$$B(B(k,\tfrac{\varepsilon}{\|B\|}))\subset B(Bk,\varepsilon)$$

Using this, the total boundedness of $T(\overline{B}(0,1))$ implies $BT(\overline{B}(0,1))$ is totally bounded. □

$\boxed{\text{Def}}$ A bounded operator $T: H\to K$ is <u>finite rank</u> if $\operatorname{ran} T$ is finite dimensional. The set of finite rank operators is denoted $FR(H,K)$, and $FR(H) := FR(H,H)$.

$\boxed{\text{Prop}}$
1. $FR(H,K)\subset K(H,K)$
2. $FR(H,K)$ is a vector subspace of $K(H,K)$.
3. For $T\in FR(H,K)$, $T^*\in FR(H,K)$
4. For $T\in FR(H,K)$, $A\in B(H)$, and $B\in B(K)$, $TA, BT\in FR(H,K)$

<u>Proof</u> (1) This follows from Exercise 1 on Homework 2.

(3) $\operatorname{ran}(T^*)\perp \ker(T)$ implies $\operatorname{ran}(T^*)\cap\ker(T)=\{0\}$. Consequently $T|_{\operatorname{ran}(T^*)}$ is injective. In particular T sends a linearly independent set in $\operatorname{ran}(T^*)$ to a linearly independent set in $\operatorname{ran}(T)$. Hence $\dim(\operatorname{ran}(T^*))\le\dim(\operatorname{ran}(T))<\infty$.
(Note that since $T|_{\operatorname{ran}(T^*)}$ is a between finite dimensional vector spaces, we then have $\dim(\operatorname{ran}(T^*)) = \dim(\operatorname{ran}(T))$ by linear algebra.)

(2), (4): Exercises. □

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$\boxed{\text{Thm}}$ For $T\in B(H,K)$, the following are equivalent.
1. T is compact.
2. T^* is compact.
3. There is a sequence $(T_n)_{n\in\mathbb{N}}\subset FR(H,K)$ s.t. $\|T-T_n\|\to 0$.

Proof (3) $\Rightarrow$ (1) & (2): Since $\mathcal{FR}(\mathcal{H},\mathcal{K}) \subset K(\mathcal{H},\mathcal{K})$, $T \in K(\mathcal{H},\mathcal{K})$ as the norm limit of compact operators. Also $\|T^* - T_n^*\| = \|T - T_n\|$ and $T_n^* \in \mathcal{FR}(\mathcal{K},\mathcal{H})$ by the previous proposition. Thus $T^* \in K(\mathcal{K},\mathcal{H})$ as well.

(1) $\Rightarrow$ (3): Since $T(\overline{B}(0,1))$ is totally bounded it is separable (Exercise: verify). Let $D \subset T(\overline{B}(0,1))$ be a countable, dense subset. Then

$$\mathbb{N}\cdot D = \{ n\cdot k : n \in \mathbb{N},\ k \in D\}$$

is countable and dense in $\operatorname{ran}(T)$. Indeed, for $h \in \mathcal{H}$ and $\varepsilon > 0$ $\exists k \in D$ s.t.

$$\left\| T\left(\frac{h}{\lceil \|h\| \rceil + 1}\right) - k \right\| < \frac{\varepsilon}{\lceil \|h\| \rceil + 1} \quad \Rightarrow \quad \| T(h) - (\lceil \|h\| \rceil + 1) k \| < \varepsilon.$$

It then follows that $\mathcal{M} := \overline{\operatorname{ran}(T)}$ is separable. Thus $\dim \mathcal{M}$ is countable. If $\dim \mathcal{M} < \infty$, then $T \in \mathcal{FR}(\mathcal{H},\mathcal{K})$ and we take $T_n = T$ $\forall n \in \mathbb{N}$. Otherwise, let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis for $\mathcal{M}$. Define $P_n := P_{\operatorname{span}\{e_1,\ldots,e_n\}}$ and $T_n := P_n T \in \mathcal{FR}(\mathcal{H},\mathcal{K})$.

Observe that for $h \in \mathcal{H}$

$$(*)\quad \|Th - T_n h\|^2 = \|(1-P_n)Th\|^2 = \sum_{k\in\mathbb{N}} |\langle (1-P_n)Th, e_k\rangle|^2 = \sum_{k>n} |\langle Th, e_k\rangle|^2 \xrightarrow{n\to\infty} 0$$

We will use this to prove $\|T - T_n\| \to 0$. Let $\varepsilon > 0$. Since $T(B(0,1))$ is totally bounded, $\exists h_1,\ldots,h_d \in \overline{B}(0,1)$ s.t.

$$T(\overline{B}(0,1)) \subset \bigcup_{j=1}^{d} B(Th_j, \tfrac{\varepsilon}{3}).$$

Using $(*)$ we can find $N \in \mathbb{N}$ s.t. $\forall n \geq N$

$$\|Th_j - T_n h_j\| < \frac{\varepsilon}{3} \qquad j = 1,\ldots,d.$$

For $h \in \mathcal{H}$ with $\|h\| \leq 1$, let j be s.t. $T(h) \in B(Th_j, \frac{\varepsilon}{3})$. Then we have

$$\begin{aligned}\|(T - T_n)h\| &\leq \|Th - Th_j\| + \|Th_j - T_n h_j\| + \|T_n h_j - T_n h\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \|P_n(Th_j - Th)\| < \varepsilon.\end{aligned}$$

Consequently $\|T - T_n\| < \varepsilon$ $\forall n \geq N$.

(2) $\Rightarrow$ (3): The same proof as above with T and T^* swapped yields a sequence $(T_n)_{n\in\mathbb{N}} \subseteq \mathcal{FR}(\mathcal{H},\mathcal{K})$ converging to T^* in norm. But then the previous prop implies $(T_n^*) \subseteq \mathcal{FR}(\mathcal{H},\mathcal{K})$, and

$$\|T - T_n^*\| = \|T^* - T_n\| \to 0.$$

$\square$

- In the proof of (1) $\Rightarrow$ (3) we established the following:

Cor For $T \in K(\mathcal{H},\mathcal{K})$, $\overline{\operatorname{ran}(T)}$ is separable. If $\{e_n : n \in \mathbb{N}\}$ is an orthonormal basis for $\overline{\operatorname{ran}(T)}$ and $P_n := P_{\operatorname{span}\{e_1,\ldots,e_n\}}$ then $\|P_n T - T\| \to 0$.

Prop Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{e_n : n \in \mathbb{N}\}$. Let $\{d_n : n \in \mathbb{N}\} \subseteq \mathbb{F}$ with

$$M := \sup\{|d_n| : n \in \mathbb{N}\} < \infty.$$

Then the map $e_n \mapsto d_n e_n$ extends to some $A \in \mathcal{B}(\mathcal{H})$ with $\|A\| = M$. Moreover, A is compact iff $\lim_{n\to\infty} d_n = 0$.

Proof Exercise 8 on Homework 2 shows the existence of A. Let $P_n := P_{\mathrm{span}\{e_1,\dots,e_n\}}$, then

$$AP_n e_k = \begin{cases} d_k e_k & \text{if } k \le n \\ 0 & \text{otherwise} \end{cases}$$

Thus $\mathrm{ran}(AP_n) = \mathrm{span}\{e_1,\dots,e_n\}$ and so $AP_n \in \mathcal{FR}(\mathcal{H})$. Also

$$(A - AP_n)e_k = \begin{cases} 0 & \text{if } k \le n \\ d_k e_k & \text{otherwise} \end{cases}$$

so that $\|A - AP_n\| = \sup\{|d_k| : k > n\}$ by Homework 2 again. So if $\lim_n d_n = 0$, $\|A - AP_n\| \to 0$ and therefore A is compact by the previous theorem. Conversely, if A is compact then $\|A - P_n A\| \to 0$ by the Corollary. But $P_n A = AP_n$, and so

$$0 = \lim_{n\to\infty} \|A - AP_n\| = \limsup_{n\to\infty} |d_n| \implies \lim_{n\to\infty} d_n = 0. \qquad \square$$

Prop Let (X, Ω, μ) be a measure space and $k \in L^2(X\times X, \Omega\times\Omega, \mu\times\mu)$. Then

$$(Kf)(x) := \int_X k(x,y) f(y)\, d\mu(y)$$

defines a compact operator K with $\|K\| \le \|k\|_2$.

- Before proving this proposition, we need a lemma (whose proof is left as exercise):

Lemma If $\{e_i : i \in I\}$ is an orthonormal basis for $L^2(X, \Omega, \mu)$, set

$$\phi_{ij}(x,y) := e_i(x)\overline{e_j(y)} \qquad i,j \in I,\ x,y \in X.$$

Then $\{\phi_{ij} : i,j \in I\}$ is an orthonormal set in $L^2(X\times X, \Omega\times\Omega, \mu\times\mu)$. With k and K as in the above proposition, $\langle k, \phi_{ij}\rangle = \langle Ke_i, e_j\rangle$.

Proof of Prop. We first show $Kf \in L^2(X, \Omega, \mu)$:

$$\int_X \left| \int_X k(x,y) f(y)\, d\mu(y)\right|^2 d\mu(x) \le \int_X \left(\int_X |k(x,y)|^2 d\mu(y)\right) \cdot \|f\|_2^2 \, d\mu(x) = \|k\|_2^2 \cdot \|f\|_2^2.$$

Thus $Kf \in L^2(X, \Omega, \mu)$ with $\|Kf\|_2 \le \|k\|_2 \|f\|_2$. This also shows K is bounded with $\|K\| \le \|k\|_2$.

Now, let $\{e_i : i \in I\}$ be an orthonormal basis for $L^2(X, \Omega, \mu)$ and define ϕ_{ij} as in the lemma. Then

$$* \quad \infty > \|k\|_2^2 \ge \sum_{i,j\in I} |\langle k, \phi_{ij}\rangle|^2 = \sum_{i,j\in I} |\langle Ke_i, e_j\rangle|^2$$

It follows that $\langle K e_i, e_j\rangle \neq 0$ for at most countably many pairs (i,j). Enumerate the i and j appearing in these pairs and relabel the corresponding e_i's as $\{e_n : n\in\mathbb{N}\}$. Set $\psi_{n,m}(x,y) := e_n(x)\overline{e_m(y)}$ and let $P_n := P_{\text{span}\{e_1,\ldots,e_n\}}$. Consider

$$K_n := KP_n + P_nK - P_nKP_n \in FR(L^2(X,\Omega,\mu))$$

Note that

$$K - K_n = (1-P_n)K(1-P_n)$$

It suffices to show $\|K-K_n\| \to 0$. Let $f\in L^2(X,\Omega,\mu)$ with $\|f\|_2\le 1$. Set $\alpha_j := \langle f, e_j\rangle$ for $j\in I$ so that $f = \sum \alpha_j e_j$. We have

$$\|Kf - K_nf\|^2 = \sum_{i\in I} |\langle Kf - K_nf, e_i\rangle|^2 = \sum_{i\in I}\Big|\sum_{j\in I}\alpha_j\langle (K-K_n)e_j, e_i\rangle\Big|^2$$

$$= \sum_{\ell\in\mathbb{N}}\Big|\sum_{m\in\mathbb{N}}\alpha_m\langle (K-K_n)e_m, e_\ell\rangle\Big|^2$$

$$\le \sum_{\ell\in\mathbb{N}}\Big(\sum_{m\in\mathbb{N}}|\alpha_m|^2\Big)\Big(\sum_{m\in\mathbb{N}}|\langle (K-K_n)e_m, e_\ell\rangle|^2\Big)$$

$$\le \|f\|_2 \sum_{\ell,m\in\mathbb{N}}|\langle (1-P_n)K(1-P_n)e_m, e_\ell\rangle|^2$$

$$\le \sum_{\ell,m\in\mathbb{N}}|\langle K(1-P_n)e_m, (1-P_n)e_\ell\rangle|^2$$

$$= \sum_{\ell,m>n}|\langle Ke_m, e_\ell\rangle|^2$$

Thus

$$\|K-K_n\| \le \sum_{\ell,m\ge n}|\langle Ke_m, e_\ell\rangle|^2$$

and by $(*)$ this tends to zero as $n\to\infty$. $\square$

Ex Recall the Volterra operator had kernel

$$k(x,y) = 1_{\{(x,y):\, y<x\}} \in L^2([0,1]\times[0,1], m\times m)$$

Therefore it is compact. $\square$

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Def For $A\in B(\mathcal{H})$, $\alpha\in\mathbb{F}$ is an <u>eigenvalue</u> of A if $\ker(A-\alpha)\neq\{0\}$. Equivalently, if there exists $h\in\mathcal{H}\setminus\{0\}$ such that $Ah = \alpha h$. In this case, h is called an <u>eigenvector</u> of A (with eigenvalue α). The <u>point spectrum</u> of A is the set of all eigenvalues of A and is denoted $\sigma_p(A)$. We say A is <u>diagonalizable</u> if there exists an orthonormal basis of eigenvectors of A.

Ex 1 Let $\mathcal{E}$ be an orthonormal basis for a Hilbert space $\mathcal{H}$. Let $\{\alpha_e : e \in \mathcal{E}\} \subseteq \mathbb{F}$ satisfy $\sup\{|\alpha_e| : e \in \mathcal{E}\} < \infty$. By Exercise 8 on Homework 2, there exists $A \in B(\mathcal{H})$ such that $Ae = \alpha_e e$ $\forall e \in \mathcal{E}$. Thus A is diagonalizable and $\sigma_p(A) = \{\alpha_e : e \in \mathcal{E}\}$. For $\alpha \in \sigma_p(A)$, set $\mathcal{E}_\alpha := \{e \in \mathcal{E} : \alpha_e = \alpha\}$. Then $h \in \mathcal{H}$ is an eigenvector of A with eigenvalue α iff $h \in \overline{\operatorname{span}}\, \mathcal{E}_\alpha$.

2 For V the Volterra operator, $\sigma_p(V) = \emptyset$. (Homework 3)

3 Let (X, Ω, μ) be a σ-finite measure space and let $\phi \in L^\infty(X, \Omega, \mu)$. For $\lambda \in \mathbb{C}$ define

$$E_\lambda := \{x \in X : \phi(x) = \lambda\}.$$

Then $\lambda \in \sigma_p(M_\phi)$ iff $\mu(E_\lambda) > 0$. Indeed, let $S \subset E_\lambda$ be such that $0 < \mu(S) < \infty$. Then $1_S \in L^2(X, \Omega, \mu)$ and

$$M_\phi 1_S = \lambda 1_S$$

So $\lambda \in \sigma_p(M_\phi)$. Conversely, if $\lambda \in \sigma_p(T)$ and $f \in L^2(X, \Omega, \mu)$ satisfies $M_\phi f = \lambda f$, then $(\phi(x) - \lambda) f(x) = 0$ for a.e. $x \in X$. If $\mu(E_\lambda) = 0$, then $f = 0$ μ-a.e. $\square$

Rem For $A \in M_n(\mathbb{C})$, $\sigma_p(A)$ corresponds to the roots of its characteristic polynomial which always exist. Contrast this with 2 above.

Prop For $T \in K(\mathcal{H})$, if $\lambda \in \sigma_p(T) \setminus \{0\}$, then $\ker(T - \lambda)$ is finite dimensional.

Proof Let $\mathcal{E}$ be an orthonormal basis for $\ker(T - \lambda)$. For $e, e' \in \mathcal{E}$ with $e \neq e'$ we observe

$$(*) \quad \|Te - Te'\| = \|\lambda e - \lambda e'\| = |\lambda| \sqrt{2}$$

Now, if $\mathcal{E}$ is infinite then we can find $\{e_n : n \in \mathbb{N}\} \subset \mathcal{E}$ with $e_n \neq e_m$ for $n \neq m$. Since T is compact, $\{Te_n : n \in \mathbb{N}\}$ contains a convergent subsequence. But this is impossible by $(*)$. So $\mathcal{E}$ must be finite. $\square$

Rem Note that $\lambda \neq 0$ was essential in our proof. It is in fact necessary: $0 \in K(\mathcal{H})$, and $\ker(0 - 0) = \mathcal{H}$.

The next result gives a way to show certain operators have at least one eigenvalue

Prop Let $T \in K(\mathcal{H})$. If $\lambda \in \mathbb{F} \setminus \{0\}$ satisfies

$$\inf_{\|h\| = 1} \|(T - \lambda) h\| = 0$$

then $\lambda \in \sigma_p(T)$.

Proof Let $(h_n)_{n \in \mathbb{N}}$ be a sequence of unit vectors satisfying

$$\|(T - \lambda) h_n\| \to 0.$$

Since T is compact, $\exists f \in \mathcal{H}$ such that $\|Th_{n_k} - f\| \to 0$ for some subsequence $(h_{n_k})_{k\in\mathbb{N}}$. Note that

$$h_{n_k} = \frac{1}{\lambda}\left[Th_{n_k} - (T-\lambda)h_{n_k} \right] \longrightarrow \lambda^{-1} f$$

Since $\|h_{n_k}\| = 1$, we have

$$\|f\| = \lambda \neq 0 .$$

Also this implies $Th_{n_k} \to T(\lambda^{-1} f) = \lambda^{-1} T f$. But by construction $Th_{n_k} \to f$. Thus $\lambda^{-1} T f = f$ or $Tf = \lambda f$. Since $f \neq 0$, this means $\lambda \in \sigma_p(T)$. □

Cor Let $T \in K(\mathcal{H})$. If $\lambda \neq 0$ is a scaler such that $\lambda \notin \sigma_p(T)$ and $\bar{\lambda} \notin \sigma_p(T^*)$, then $\operatorname{ran}(T-\lambda) = \mathcal{H}$ and $(T-\lambda)^{-1} \in \mathcal{B}(\mathcal{H})$.

Proof $\lambda \notin \sigma_p(T)$ implies, by the preceeding proposition, that $\exists c > 0$ s.t.

$$* \quad \|(T-\lambda)h\| \geq c\|h\| \qquad \forall h \in \mathcal{H}$$

Let $f \in \overline{\operatorname{ran}(T-\lambda)}$. Then $\exists (h_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ such that $(T-\lambda)h_n \to f$. Using $(*)$ we have

$$\|h_n - h_m\| \leq \frac{1}{c} \|(T-\lambda)(h_n - h_m)\|$$

and so $(h_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $h = \lim_{n\to\infty} h_n$, then $(T-\lambda)h = f$. Thus $\operatorname{ran}(T-\lambda)$ is closed. But then since $\bar{\lambda} \notin \sigma_p(T^*)$

$$\operatorname{ran}(T-\lambda) = \ker(T^* - \bar{\lambda})^{\perp} = \{0\}^{\perp} = \mathcal{H}.$$

Thus $T-\lambda$ is a bijection. Also $(*)$ says precisely that $\|(T-\lambda)^{-1}\| = \frac{1}{c}$. □

• We will eventually see that $\bar{\lambda} \notin \sigma_p(T^*)$ follows from $\lambda \notin \sigma_p(T)$ and $\lambda \neq 0$.

II.5 Diagonalization of Self-Adjoint Compact Operators

- In linear algebra we learn that all self-adjoint matrices are diagonalizable. It turns out this is true for normal $A \in B(H)$ (even when H is infinite dimensional!), but we aren't ready yet to see this theorem (called the Spectral Theorem). We can, however, prove the following special case

Thm (Spectral Theorem for Self-Adjoint Compact Operators)
If $T = T^* \in K(H)$, then $\sigma_p(T) \subseteq \mathbb{R}$ and is countable. Let $\{\lambda_1, \lambda_2, \ldots\}$ be the distinct eigenvalues of T, and let $P_n := P_{\ker(T-\lambda_n)}$. Then $\{P_n\}_{n\in\mathbb{N}}$ are pairwise orthogonal and

$$T = \sum_{n=1}^{\infty} \lambda_n P_n$$

where the series converges in operator norm.

Ex Let $A = A^* \in M_n(\mathbb{C})$ with distinct eigenvalues $\lambda_1, \ldots, \lambda_d$ and multiplicities $m_1, \ldots, m_d$. We know from linear algebra that $\exists$ unitary $U \in M_n(\mathbb{C})$ s.t.

$$A = U \begin{pmatrix} \underbrace{\lambda_1 \cdots \lambda_1}_{m_1} & & 0 \\ & \ddots & \\ 0 & & \underbrace{\lambda_d \cdots \lambda_d}_{m_d} \end{pmatrix} U^*$$

For each $j = 1, \ldots, d$ let

$$E_j := \begin{pmatrix} 0 \ddots 0 & & 0 \\ & \underbrace{1 \cdots 1}_{m_j} & \\ 0 & & 0 \ddots 0 \end{pmatrix}$$

Then E_j and $P_j := U E_j U^*$ are projections. Also

$$A = U \sum_{j=1}^{d} \lambda_j E_j \, U^* = \sum_{j=1}^{d} \lambda_j P_j$$

Note that if $\lambda_j = 0$, then $\lambda_j P_j = 0$ and can be excluded from the above sum. $\square$

- We require a few additional results before proving the above theorem.

Prop For $A \in B(H)$ normal and $\lambda \in \mathbb{F}$, $\ker(A-\lambda) = \ker(A-\lambda)^*$ and is reducing for A.

Proof $A - \lambda$ is normal by virtue of A being normal. So by our characterization of normality in section II.2 we have

$$\|(A-\lambda)h\| = \|(A-\lambda)^* h\| \qquad \forall h \in H.$$

Consequently $\ker(A-\lambda) = \ker(A-\lambda)^*$. If $h \in \ker(A-\lambda)$, then $Ah = \lambda h \in \ker(A-\lambda)$. So $\ker(A-\lambda)$ is invariant for A. Also $h \in \ker(A-\lambda)^*$ so $A^* h = \bar{\lambda} h \in \ker(A-\lambda)^*$, and $\ker(A-\lambda) = \ker(A-\lambda)^*$ is invariant for A^*. By a characterization from Section II.3 we have that $\ker(A-\lambda)$ is reducing for A. $\square$

Prop For $A \in B(H)$ normal, if $\lambda, \mu \in \sigma_p(A)$ are distinct then $\ker(A-\lambda) \perp \ker(A-\mu)$.

Proof Let $g \in \ker(A-\mu)$. By the previous proposition $g \in \ker(A-\mu)^*$ and so $A^*g = \bar{\mu} g$. Now, for $f \in \ker(A-\lambda)$ we have

$$\frac{\lambda}{\mu}\langle f, g\rangle = \frac{1}{\mu}\langle Af, g\rangle = \frac{1}{\mu}\langle f, A^*g\rangle = \frac{1}{\mu}\langle f, \bar{\mu} g\rangle = \langle f, g\rangle$$

or $(\frac{\lambda}{\mu} - 1)\langle f, g\rangle = 0$. Since $\lambda \neq \mu$, we must have $\langle f, g\rangle = 0$. $\square$

Prop For $A \in B(H)$ self-adjoint, $\sigma_p(A) \subseteq \mathbb{R}$

Proof A is in particular normal, so if $\lambda \in \sigma_p(A)$ then

$$\ker(A-\lambda) = \ker(A-\lambda)^* = \ker(A-\bar{\lambda})$$

For $h \in \ker(A-\lambda)\setminus\{0\}$ we then have $\lambda h = Ah = \bar{\lambda} h$, or $(\lambda - \bar{\lambda})h$. So $\lambda = \bar{\lambda}$ since $h \neq 0$. $\square$

- To jump start the proof of the spectral theorem for $T = T^* \in K(H)$, we have to show that if $T \neq 0$ then $\sigma_p(T) \neq \emptyset$.

Lemma If $T \in K(H)$ is self-adjoint, then either $\|T\| \in \sigma_p(T)$ or $-\|T\| \in \sigma_p(T)$.

Proof If $T = 0$ then we are done. Suppose $T \neq 0$. In section II.2 we showed

$$\|T\| = \sup_{\|h\|=1} |\langle Th, h\rangle|$$

So let $(h_n)_{n\in\mathbb{N}} \subset H$ be such that $\|h_n\| = 1$ and $|\langle Th_n, h_n\rangle| \longrightarrow \|T\|$. Note that

$$\overline{\langle Th_n, h_n\rangle} = \langle h_n, Th_n\rangle = \langle Th_n, h_n\rangle \in \mathbb{R}$$

so by passing to a subsequence we get $\langle Th_n, h_n\rangle \longrightarrow \lambda$ where $\lambda \in \{-\|T\|, \|T\|\}$.
Observe that

$$0 \leq \|(T-\lambda)h_n\|^2 = \|Th_n\|^2 - 2\lambda\langle Th_n, h_n\rangle + \lambda^2 \leq 2\lambda^2 - 2\lambda\langle Th_n, h_n\rangle \to 0$$

Consequently

$$\inf_{\|h\|=1} \|(T-\lambda)h\| = 0$$

and so $\lambda \in \sigma_p(T)$ by a proposition from Section II.4 $\square$

Proof of Spectral Theorem for Self-Adjoint Compact Operators

By the lemma $\exists\, \lambda_1 \in \sigma_p(T)$ s.t. $|\lambda_1| = \|T\|$. Set

$$\mathcal{E}_1 := \ker(T-\lambda_1)$$
$$P_1 := P_{\mathcal{E}_1}$$

Since T is normal, $\mathcal{E}_1$ is reducing for T and so $TP_1 = P_1 T$. This implies

$$T(1-P_1) = (1-P_1)T(1-P_1)$$

Define $T_2 := T(1-P_1)$, which is compact and self-adjoint. Using the lemma $\exists\, \lambda_2 \in \sigma_p(T_2)$ s.t. $|\lambda_2| = \|T_2\|$. We claim $\ker(T_2 - \lambda_2) = \ker(T-\lambda_2)$. Indeed, for $h \in \ker(T_2 - \lambda_2)$:

$$\lambda_2 h = T(1-P_1)h = (1-P_1)T(1-P_1)h \in \mathcal{E}_1^\perp$$

So $(1-P_1)h = h$ and consequently

$$Th = T(1-P_1)h = T_2 h = \lambda_2 h$$

That is, $\ker(T_2 - \lambda_2) \subseteq \ker(T-\lambda_2)$. We also showed $\ker(T_2-\lambda_2) \subseteq \mathcal{E}_1^\perp$, so we must have $\lambda_2 \neq \lambda_1$ since $\mathcal{E}_1 \cap \mathcal{E}_1^\perp = \{0\}$. Conversely, let $h \in \ker(T-\lambda_2)$. Then $h \in \mathcal{E}_1^\perp$ by an earlier proposition, and so

$$T_2 h = T(1-P_1)h = Th = \lambda_2 h.$$

Thus $\ker(T_2 - \lambda_2) = \ker(T-\lambda_2)$ as claimed. Also note that $|\lambda_2| = \|T_2\| \leq \|T\| = |\lambda_1|$.
Set

$$\mathcal{E}_2 := \ker(T-\lambda_2)$$
$$P_2 := P_{\mathcal{E}_2}.$$

We inductively obtain a sequence $(\lambda_n)_{n\in\mathbb{N}} \subseteq \sigma_p(T)$ such that 2/5

(i) $|\lambda_1| \geq |\lambda_2| \geq \cdots$

(ii) if $\mathcal{E}_n = \ker(T-\lambda_n)$ and $P_n = P_{\mathcal{E}_n}$ then $|\lambda_{n+1}| = \|T(1-\sum_{i=1}^n P_i)\|$

By (i) $\alpha := \lim|\lambda_n|$ exists. We claim $\alpha = 0$. Let $e_n \in \mathcal{E}_n$ with $\|e_n\| = 1$. By compactness of T $\exists h \in \mathcal{H}$ and a subsequence $(e_{n_j})_{j\in\mathbb{N}}$ s.t. $\|Te_{n_j} - h\| \to 0$.
But for $j \neq \ell$

$$\|Te_{n_j} - Te_{n_\ell}\|^2 = \|\lambda_{n_j} e_{n_j} - \lambda_{n_\ell} e_{n_\ell}\|^2 = |\lambda_{n_j}|^2 + |\lambda_{n_\ell}|^2 \geq 2\alpha^2$$

Since (Te_{n_j}) is a Cauchy sequence, we must have $\alpha = 0$.
Now, for $n \in \mathbb{N}$ we claim

$$* \quad T - \sum_{j=1}^n \lambda_j P_j = T(1 - \sum_{j=1}^n P_j)$$

For $h \in \mathcal{E}_k$, $1 \leq k \leq n$ we have

$$(T - \sum_{j=1}^n \lambda_j P_j)h = Th - \lambda_k h = 0 = T0 = T(1 - \sum_{j=1}^n P_j)h$$

Thus $(*)$ holds on $\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_n$. For $h \in (\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_n)^\perp$ we have

$$(T - \sum_{j=1}^n \lambda_j P_j)h = Th - 0 = Th = T(1 - \sum_{j=1}^n P_j)h.$$

Thus $(*)$ holds on all of $\mathcal{H}$. But then (ii) gives

$$\lim_{n\to\infty} \|T - \sum_{j=1}^n \lambda_j P_j\| = \lim_{n\to\infty} \|T(1 - \sum_{j=1}^n P_j)\| = \lim_{n\to\infty} |\lambda_{n+1}| = 0.$$

In particular, this implies $\sigma_p(T) \subset \{0, \lambda_1, \lambda_2, \ldots\} \subseteq \mathbb{R}$ and is countable. $\square$

Cor With $T, \lambda_n, \mathcal{E}_n$, and P_n as in the theorem

1. $\ker T = (\bigoplus_{n=1}^\infty \mathcal{E}_n)^\perp$
2. $P_n \in \mathcal{FR}(\mathcal{H})$ $\forall n \in \mathbb{N}$
3. $\|T\| = \sup_n |\lambda_n|$ and $\lim_n \lambda_n = 0$ if $\sigma_p(T)$ is infinite.

Proof (1): Since $\mathcal{E}_n \perp \mathcal{E}_m$ for $n \neq m$ we have

$$\|Th\|^2 = \|\sum_{n=1}^\infty \lambda_n P_n h\|^2 = \sum_{n=1}^\infty |\lambda_n|^2 \|P_n h\|^2$$

So $h \in \ker T$ iff $h \in \mathcal{E}_n^\perp$ for all n.

(2): $\operatorname{ran} P_n = \mathcal{E}_n = \ker(T-\lambda_n)$ and since $T \in K(\mathcal{H})$ this finite dimensional by a proposition from Section II.4.

(3): These were both shown in the proof of the theorem, but we can also prove them using (1). Let $\mathcal{K} := \overline{\operatorname{ran}}\, T$, which is invariant for T. Since $T = T^*$, it is also invariant for T^* and therefore is reducing for T. Let $P := P_{\mathcal{K}}$, and consider

$$PTP : \mathcal{K} \to \mathcal{K}$$

By (1), $\bigoplus_{n=1}^{\infty} \mathcal{E}_n = (\ker T)^{\perp} = \mathcal{K}$. Let $\{e_j^{(n)} : 1 \le j \le N_n\}$ be an orthonormal basis for $\mathcal{E}_n$, so that

$$\{e_j^{(n)} : n \in \mathbb{N},\ 1 \le j \le N\}$$

is an orthonormal basis for $\mathcal{K}$. Since these are all eigenvectors of T, T is diagonalizable. Thus $\|T\| = \sup_n |\lambda_n|$. Since T is compact we also have $\lim |\lambda_n| \to 0$ by a proposition from Section II.4. $\square$

$\boxed{\text{Cor}}$ Let $T = T^* \in K(\mathcal{H})$, then $\exists\, (\mu_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ and an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ for $(\ker T)^{\perp}$ such that

$$Th = \sum_{n=1}^{\infty} \mu_n \langle h, e_n \rangle e_n \qquad \forall h \in \mathcal{H}$$

- Here $\{\mu_n : n \in \mathbb{N}\} = \sigma_p(T) \setminus \{0\}$ with repetitions according to $\dim \mathcal{E}_n$.

$\boxed{\text{Cor}}$ If $T = T^* \in K(\mathcal{H})$ and $\dim(\ker T) < \infty$, then $\mathcal{H}$ is separable. In particular, if T is injective.

Proof The previous corollary implies $(\ker T)^{\perp}$ is separable and $\mathcal{H} = \ker T \oplus (\ker T)^{\perp}$ $\square$

II.7 Spectral Theorem and Functional Calculus for Compact Normal Operators

• In this section we will extend the spectral theorem from last section to compact normal operators. We will also explore the "functional calculus" for such operators. You have already seen a version of this in Exercise 9 of Homework 2 for power series.

Prop Let $\{P_i : i \in I\}$ be a family of pairwise orthogonal projections in $B(\mathcal{H})$. For $h \in \mathcal{H}$
$$\sum_{i \in I} P_i h$$
converges and equals $P_K h$ where
$$K = \bigoplus_{i \in I} \operatorname{ran} P_i .$$

Proof Homework 3. $\square$

• In light of the above proposition, we write
$$P_K = \sum_{i \in I} P_i$$
However, $\sum_{i \in I} P_i$ does not converge to P_K in operator norm unless I is finite. It merely converges pointwise on $\mathcal{H}$ (i.e. with respect to the strong operator topology)

Def A partition of unity on $\mathcal{H}$ is a family $\{P_i : i \in I\}$ of pairwise orthogonal projections such that $\bigoplus_{i \in I} \operatorname{ran} P_i = \mathcal{H}$. We write $1 = \sum_{i \in I} P_i$ to denote this.

Prop $A \in B(\mathcal{H})$ is diagonalizable iff there exists a partition of unity on $\mathcal{H}$ $\{P_i : i \in I\}$ and scalars $\{\alpha_i : i \in I\}$ satisfying $\sup_{i \in I} |\alpha_i| < \infty$ such that $AP_i = \alpha_i P_i$.

Proof ($\Longrightarrow$) Let $\mathcal{E}$ be an orthonormal basis of eigenvectors of A. For $\lambda \in \sigma_p(A)$, let $\mathcal{E}_\lambda := \mathcal{E} \cap \ker(A - \lambda)$. Then
$$\mathcal{E} = \bigcup_{\lambda \in \sigma_p(A)} \mathcal{E}_\lambda$$
and consequently if $P_\lambda := P_{\overline{\operatorname{span}}\, \mathcal{E}_\lambda}$, it follows that $\{P_\lambda : \lambda \in \sigma_p(A)\}$ is a partition of unity. Moreover, $Ah = \lambda h$ $\forall\, h \in \overline{\operatorname{span}}\, \mathcal{E}_\lambda = \operatorname{ran} P_\lambda$. Thus $AP_\lambda = \lambda P_\lambda$ $\forall \lambda \in \sigma_p(A)$. Lastly, $\forall \lambda \in \sigma_p(A)$ if $e \in \mathcal{E}_\lambda$ we have
$$|\lambda| = \|\lambda e\| = \|Ae\| \le \|A\|$$
so $\sup_{\lambda \in \sigma_p(A)} |\lambda| \le \|A\|$.

($\Longleftarrow$) For each $i \in I$, let $\mathcal{E}_i$ be an orthonormal basis for $\operatorname{ran} P_i$. Then
$$\mathcal{H} = \bigoplus_{i \in I} \operatorname{ran} P_i$$
implies $\mathcal{E} := \bigcup_{i \in I} \mathcal{E}_i$ is an orthonormal basis for $\mathcal{H}$. Furthermore, $\forall e \in \mathcal{E}_i$ we have
$$Ae = AP_i e = \alpha_i P_i e = \alpha_i e$$
So $\mathcal{E}$ consists of eigenvectors of A. $\square$

- If A, $\{P_i : i \in I\}$, and $\{\alpha_i : i \in I\}$ are as above we write
$$A = \sum_{i \in I} \alpha_i P_i$$
However, we again caution that the series on the right does not converge in operator norm to A. It just converges in the strong operator topology (Exercise).

- Note that if $\alpha_i = \alpha_j$, then $\alpha_i P_i + \alpha_j P_j = \alpha_i (P_i + P_j)$ and so we can replace P_i and P_j with their sum. In this way we may assume the α_i are distinct.

- If $A = \sum_{i \in I} \alpha_i P_i$, note that $AP_i = \alpha_i P_i$ implies
$$AP_i = P_i A P_i \overset{*}{=} P_i A \qquad (* \text{ Exercise})$$
Consequently $\operatorname{ran} P_i$ is reducing for A. Furthermore, this implies
$$A^* P_i = (P_i A)^* = (AP_i)^* = (\alpha_i P_i)^* = \bar{\alpha}_i P_i$$
So that $A^* = \sum_{i \in I} \bar{\alpha}_i P_i$. Consequently A is normal since
$$A^* A P_i = |\alpha_i|^2 P_i = A A^* P_i \qquad \forall i \in I.$$

Prop If $A = \sum_{i \in I} \alpha_i P_i$ is diagonalizable and the α_i are all distinct, then $B \in \mathcal{B}(\mathcal{H})$ satisfies $AB = BA$ iff $\operatorname{ran} P_i$ is reducing for B for every $i \in I$.

Proof ($\Longrightarrow$) Suppose $AB = BA$. Then for $i, j \in I$ we have
$$\alpha_j P_j B P_i = A P_j B P_i = P_j B A P_i = \alpha_i P_j B P_i$$
or $(\alpha_j - \alpha_i) P_j B P_i = 0$. Consequently, $P_j B P_i = 0$ for $i \neq j$ since $\alpha_i \neq \alpha_j$. Now, for any $h \in \mathcal{H}$ we have
$$BP_i h = \sum_{j \in I} P_j (BP_i h) = \sum_{j \in I} (P_j B P_i) h = P_i B P_i h.$$
Thus $BP_i = P_i B P_i$, and so $\operatorname{ran} P_i$ is invariant for B. By taking adjoints we have $A^* B^* = B^* A^*$ and $A^* = \sum \bar{\alpha}_i P_i$. So the same argument implies $\operatorname{ran} P_i$ is invariant for B^*. Hence $\operatorname{ran} P_i$ is reducing for B.

($\Longleftarrow$) We have $P_i B = B P_i$ for each $i \in I$. Thus for any finite $F \subset I$
$$\Big(\sum_{i \in F} \alpha_i P_i\Big) B = B \Big(\sum_{i \in F} \alpha_i P_i\Big)$$
It follows that $\forall h \in \mathcal{H}$
$$ABh = \sum_{i \in I} \alpha_i P_i B h = B \sum_{i \in I} \alpha_i P_i h = BAh$$
so $AB = BA$. $\square$

Thm (Spectral Theorem for Compact Normal Operators)
Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$. If $T \in \mathcal{K}(\mathcal{H})$ is normal, then $\sigma_p(T)$ is countable. Let $\{\lambda_1, \lambda_2, \ldots\}$ be the distinct nonzero eigenvalues of T, and let $P_n = P_{\ker(A - \lambda_n)}$. Then $\{P_n : n \in \mathbb{N}\}$ are pairwise orthogonal and
$$T = \sum_{n=1}^{\infty} \lambda_n P_n$$
where the series converges in operator norm.

~~Proof~~ Define
$$A := \mathrm{Re}\, T = \tfrac{1}{2}(T+T^*) \quad \text{and} \quad B := \mathrm{Im}\, T = \tfrac{1}{2i}(T-T^*)$$
Then $A, B \in K(H)$ and commute since T is normal. So by the spectral theorem for compact self-adjoint operators
$$A = \sum_{j=1}^{\infty} \alpha_j Q_j$$
for distinct nonzero $\alpha_j \in \mathbb{R}$ and Q_j pairwise orthogonal projections. If we let $Q_0 = P_{\ker A}$, then $\{Q_j\}_{j=0}^{\infty}$ is a partition of unity on H. By the previous proposition, $\mathrm{ran}\, Q_j$ is reducing for B for each $j \geq 0$. Hence $Q_j B = B Q_j$. Therefore $B|_{\mathrm{ran}\, Q_j} \in K(\mathrm{ran}\, Q_j)$. Applying the spectral theorem again we obtain for each $j \geq 0$,
$$B|_{\mathrm{ran}\, Q_j} = \sum_{k=1}^{\infty} \beta_k^{(j)} Q_k^{(j)}$$
Let $Q_0^{(j)} := P_{\ker\{B|_{\mathrm{ran}\, Q_j}\}}$, we have that $\{Q_k^{(j)} : k \geq 0\}$ is a partition of unity on $\mathrm{ran}\, Q_j$. Consequently $\{Q_k^{(j)} : j, k \geq 0\}$ is a partition of unity on H. Observe that $Q_j Q_k^{(j)} = Q_k^{(j)}$. Thus
$$T Q_k^{(j)} = (A + iB) Q_k^{(j)} = A Q_j Q_k^{(j)} + i B Q_k^{(j)} = \alpha_j Q_k^{(j)} + i \beta_k^{(j)} Q_k^{(j)} = (\alpha_j + i\beta_k^{(j)}) Q_k^{(j)}.$$
It follows that $\sigma_p(T) = \{\alpha_j + i\beta_k^{(j)} : i, j \geq 0\}$ and so is countable. It remains to show that
$$T = \sum_{j,k=1}^{\infty} (\alpha_j + i\beta_k^{(j)}) Q_k^{(j)}$$
where the series converges in operator norm. Recall that $\dim(\mathrm{ran}\, Q_i) < \infty$ for $i \geq 0$. Consequently $Q_k^{(j)} = 0$ for all but finitely many k. Let $k(j)$ be s.t. $Q_k^{(j)} = 0$ for $k \geq k(j)$. Recall that
$$\| A - \sum_{j=1}^{N} \alpha_j Q_j \| = \sup_{j > N} |\alpha_j|.$$
Moreover, one can show that if $M \geq \max\{k(1), \dots, k(N_0)\}$, then for $N \geq N_0$
$$\| A - \sum_{j=1}^{N} \sum_{k=1}^{M} \alpha_j Q_k^{(j)} \| \leq \sup_{j > N_0} |\alpha_j|.$$

Similarly
$$\| B - \sum_{j=1}^{N} \sum_{k=1}^{M} \beta_j^{(k)} Q_k^{(j)} \| \leq \sup_{j > N_0} \sup_{k > M} |\beta_k^{(j)}|.$$
Since A and B are compact, we have
$$\lim_{j \to \infty} |\alpha_j| = 0 \quad \text{and} \quad \lim_{jk \to \infty} |\beta_k^{(j)}| = 0$$

Using this, the above estimates, and $T = A + iB$ one can obtain norm convergence we leave the details as an exercise. $\square$

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Cor ① $\ker(T) = \left(\bigoplus_{n=1}^{\infty} \mathrm{ran}\, P_n\right)^{\perp}$

② $P_n \in FR(H) \quad \forall n \in \mathbb{N}$.

③ $\|T\| = \sup\{|\lambda_n| : n \geq 1\}$ and $\lim_{n\to\infty} \lambda_n = 0$ if $\sigma_p(T)$ is infinite.

• The proof of the above is similar to the corresponding result for compact self-adjoint operators, so we leave it as an exercise.

• Recall that we showed any diagonalizable operator is normal. Thus we obtain:

Cor Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$. Then $T \in K(\mathcal{H})$ is normal iff it is diagonalizable.

Functional Calculus

• For the remainder of this section, let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$. Let $\ell^\infty(\mathbb{C})$ denote the set of all bounded functions $\phi: \mathbb{C} \to \mathbb{C}$.

• For $T \in K(\mathcal{H})$ normal, let
$$T = \sum_{n=1}^{\infty} \lambda_n P_n$$
where $P_n = P_{\ker(T-\lambda_n)}$ and let $P_0 = \ker(T)$. For $\phi \in \ell^\infty(\mathbb{C})$, note that
$$\sup_n |\phi(\lambda_n)|$$
is bounded. Hence
$$\phi(T) := \sum_{n=1}^{\infty} \phi(\lambda_n) P_n + \phi(0) P_0$$
defines a bounded diagonalizable operator with
$$\|\phi(T)\| = \sup \{|\phi(0)|\} \cup \{|\phi(\lambda_n)| : n \in \mathbb{N}\}$$

Thm Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$. For $T \in K(\mathcal{H})$ normal, the map
$$\ell^\infty(\mathbb{C}) \to B(\mathcal{H})$$
$$\phi \mapsto \phi(T)$$
defined above has the following properties

1. It is multiplicative in the sense that $(\phi \cdot \psi)(T) = \phi(T)\psi(T)$ for $\phi, \psi \in \ell^\infty(\mathbb{C})$.
2. If $\phi \equiv 1$ then $\phi(T) = I$ and if $\phi(z) = z$ for $z \in \sigma_p(T) \cup \{0\}$, then $\phi(T) = T$.
3. $\phi(T)^* = \bar{\phi}(T)$ where $\bar{\phi}(z) = \overline{\phi(z)}$.
4. For $A \in B(\mathcal{H})$, $AT = TA$ iff $A\phi(T) = \phi(T)A$ $\forall \phi \in \ell^\infty(\mathbb{C})$.

Proof (1): Recall that $P_n P_m = 0$ if $n \neq m$. Thus for $h \in \mathcal{H}$
$$\phi(T)\psi(T)h = \left(\sum_{n=1}^{\infty} \phi(\lambda_n) P_n + \phi(0) P_0\right)\left(\sum_{m=1}^{\infty} \psi(\lambda_m) P_m h + \psi(0) P_0 h\right)$$
$$= \sum_{n=1}^{\infty} \phi(\lambda_n)\psi(\lambda_n) P_n h + \phi(0)\psi(0) P_0 h = (\phi\psi)(T)h.$$

(2): If $\phi \equiv 1$, then
$$\phi(T) = \sum_{n=0}^{\infty} P_n + P_0 = I$$
since $\{P_0, P_1, P_2, \dots\}$ is a partition of unity. If $\phi(z) = z$ on $\sigma_p(T) \cup \{0\}$ we have

$$\phi(T) = \sum_{n=1}^{\infty} \lambda_n P_n + 0\cdot P_0 = \sum_{n=1}^{\infty} \lambda_n P_n = T.$$

(3): Since $\phi(T)$ is diagonalizable, we have:

$$\phi(T)^* = \sum_{n=1}^{\infty} \overline{\phi(\lambda_n)} P_n + \overline{\phi(0)} P_0 = \overline{\phi}(T).$$

(4) ($\Rightarrow$) We know $AT = TA$ implies $AP_n = P_n A$ for all $n \geq 0$. Thus

$$A\left(\sum_{n=1}^{N} \phi(\lambda_n) P_n + \phi(0) P_0\right) = \left(\sum_{n=1}^{N} \phi(\lambda_n) P_n + \phi(0) P_0\right) A$$

for each $N \in \mathbb{N}$. Using strong operator topology convergence yields $A\phi(T) = \phi(T)A$.
($\Leftarrow$) Using 2 we have for $\phi(z) = z$, $AT = A\phi(T) = \phi(T)A = TA$. $\square$

Def For $S \subset B(H)$, the <u>commutant</u> of S is the set

$$S' = \{A \in B(H) : AB = BA \text{ for all } B \in S\}.$$

The <u>double commutant</u> of S is $S'' := (S')'$.

- Observe that S' is closed under addition and multiplication, i.e. it is an algebra. If S closed under taking adjoints, then S' is a $*$-algebra. Also $1 \in S'$ always.

Ex $(\mathbb{C} I_H)' = B(H)$ and $B(H)' = \mathbb{C} I_H$ (Homework 3)

Thm Let H be a Hilbert space over $\mathbb{C}$. For $T \in K(H)$ normal

$$\{\phi(T) : \phi \in \ell^\infty(\mathbb{C})\} = \{T\}''$$

Proof By 4 from the previous theorem we immediately obtain $\subseteq$. Let $A \in \{T\}''$. Since $T \in \{T\}'$ (it commutes with itself of course), we have $AT = TA$. By the proposition preceding the spectral theorem we see that $E_\lambda := \ker(T - \lambda)$ is reducing for A for each $\lambda \in \sigma_p(T)$.
Fix $\lambda \in \sigma_p(T)$. For any $B_\lambda \in B(E_\lambda)$, we can extend B_λ to $B \in B(E_\lambda)$ by

$$Bh = \begin{cases} B_\lambda h & \text{if } h \in E_\lambda \\ 0 & \text{if } h \in E_\lambda^\perp \end{cases}$$

It follows that $BP_{E_\mu} = P_{E_\mu} B$ $\forall \mu \in \sigma_p(T)$. So E_μ is reducing for B for each $\mu \in \sigma_p(T)$. By the same proposition invoked earlier we obtain $BT = TB$. Consequently $AB = BA$. In particular

$$P_{E_\lambda} A P_{E_\lambda} B_\lambda = B_\lambda P_{E_\lambda} A P_{E_\lambda}.$$

Since $B_\lambda \in B(E_\lambda)$ was arbitrary, we have

$$P_{E_\lambda} A P_{E_\lambda} \in B(E_\lambda)' = \mathbb{C} P_{E_\lambda}$$

by the above example. Thus $P_{E_\lambda} A P_{E_\lambda} = \alpha_\lambda P_{E_\lambda}$. Note that $|\alpha_\lambda| = \|P_{E_\lambda} A P_{E_\lambda}\| \leq \|A\|$.
So we define $\phi \in \ell^\infty(\mathbb{C})$ by $\phi(\lambda) := \alpha_\lambda$ and it follows that $A = \phi(T)$. $\square$

Def We say $A \in \mathcal{B}(\mathcal{H})$ is positive if $\langle Ah, h\rangle \geq 0$ $\forall h \in \mathcal{H}$. We write $A \geq 0$.

Ex For $A \in M_n(\mathbb{C})$ diagonalizable, A is positive iff $\sigma_p(A) \subset [0,\infty)$.

Prop For $T \in K(\mathcal{H})$ normal, T is positive iff $\sigma_p(T) \subset [0,\infty)$.

Proof Let $T = \sum_{n=1}^{\infty} \lambda_n P_n$.

($\Rightarrow$) For $h \in P_n\mathcal{H} \setminus \{0\}$ we have

$$0 \leq \langle Th, h\rangle = \langle \lambda_n h, h\rangle = \lambda_n \|h\|^2$$

So $\lambda_n \geq 0$.

($\Leftarrow$) For $h \in \mathcal{H}$ write

$$h = h_0 + \sum_{n=1}^{\infty} h_n$$

where $h_0 \in \ker T$ and $h_n \in P_n\mathcal{H}$. Then

$$\langle Th, h\rangle = \left\langle \sum_{n=1}^{\infty} \lambda_n h_n, \sum_{m=0}^{\infty} h_m \right\rangle = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \lambda_n \langle h_n, h_m\rangle = \sum_{n=1}^{\infty} \lambda_n \|h_n\|^2 \geq 0. \quad \square$$

Thm If $T \in K(\mathcal{H})$ is self-adjoint, then there exists unique positive $A, B \in K(\mathcal{H})$ such that $T = A - B$ and $AB = BA = 0$.

Proof Since $T = T^*$, $\sigma_p(T) \subset \mathbb{R}$. Define $\phi, \psi \in \ell^\infty(\mathbb{C})$ by

$$\phi(\lambda) = \begin{cases} \lambda & \text{if } \lambda \in \sigma_p(T) \cap [0,\infty) \\ 0 & \text{otherwise} \end{cases} \qquad \psi(\lambda) = \begin{cases} -\lambda & \text{if } \lambda \in \sigma_p(T) \cap (-\infty, 0) \\ 0 & \text{otherwise} \end{cases}$$

Set $A := \phi(T)$ and $B := \psi(T)$. Note that $(\phi - \psi)(\lambda) = \lambda$ for $\lambda \in \sigma_p(T) \cup \{0\}$ and thus

$$A - B = (\phi - \psi)(T) = T.$$

Since $\phi\psi(z) = 0$, we obtain

$$AB = (\phi\psi)(T) = 0 = (\psi\phi)(T) = BA.$$

If $T = \sum_{\lambda \in \sigma_p(T)} \lambda P_\lambda$, then

$$A = \sum_{\lambda \in \sigma_p(T) \cap [0,\infty)} \lambda P_\lambda \Rightarrow \sigma_p(A) \subset [0,\infty) \Rightarrow A \geq 0$$

$$B = \sum_{\lambda \in \sigma_p(T) \cap (-\infty,0)} -\lambda P_\lambda \Rightarrow \sigma_p(B) \subset [0,\infty) \Rightarrow B \geq 0.$$

It remains to show A and B are unique. Suppose $C, D \in K(\mathcal{H})$ are positive with $T = C - D$ and $CD = DC = 0$. Then

$$CT = C(C - D) = C^2 = (C - D)C = TC$$

Similarly $DT = TD$. Then C and D are reduced by $P_\lambda\mathcal{H}$ $\forall \lambda \in \sigma_p(T)$ and $\ker(T)$. Put $P_0 := \ker(T)$. For $C_\lambda := P_\lambda C P_\lambda$ and $D_\lambda := P_\lambda D P_\lambda$, $\lambda \in \sigma_p(T) \cup \{0\}$, we have $C_\lambda D_\lambda = D_\lambda C_\lambda = 0$, $C_\lambda, D_\lambda \geq 0$ and

$$C_\lambda - D_\lambda = P_\lambda T P_\lambda = \lambda P_\lambda$$

Now, suppose $\lambda > 0$. Note that $C_\lambda D_\lambda = 0$ implies $\ker(C_\lambda) \supset \overline{\operatorname{ran} D_\lambda} = (\ker D_\lambda)^\perp$. Let $h \in (\ker D_\lambda)^\perp$. Then

$$\lambda h = \lambda P_\lambda h = \lambda(C_\lambda - D_\lambda)h = -\lambda D_\lambda h$$

So

$$\lambda \|h\|^2 = \langle \lambda h, h\rangle = \langle -\lambda D_\lambda h, h\rangle = -\lambda \langle D_\lambda h, h\rangle \leq 0$$

Since $\lambda > 0$, it must be that $\|h\| = 0$. Hence $(\ker D_\lambda)^\perp = \{0\} \Rightarrow \ker D_\lambda = P_\lambda \mathcal{H}$.
Therefore

$$D_\lambda = 0 = P_\lambda B P_\lambda$$
$$C_\lambda = \lambda P_\lambda = P_\lambda A P_\lambda$$

Similarly, for $\lambda < 0$ we can show

$$C_\lambda = 0 = P_\lambda A P_\lambda$$
$$D_\lambda = -\lambda P_\lambda = P_\lambda B P_\lambda$$

Finally, for $\lambda = 0$ we have $0 = C_0 - D_0$ or $C_0 = D_0$. But then $0 = C_0 D_0 = C_0^2$ so that for any $h \in \ker(T)$

$$\|C_0 h\|^2 = \langle C_0 h, C_0 h\rangle = \langle C_0^2 h, h\rangle = 0$$

Hence

$$0 = C_0 = D_0 = P_0 A P_0 = P_0 B P_0.$$

Thus $C = A$ and $D = B$. $\square$

- we usually write $T_+ := A$, $T_- := B$.

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Thm If $T \in K(\mathcal{H})$ is positive, then there exists a unique positive $A \in K(\mathcal{H})$ such that $A^2 = T$.

Proof $T \geq 0$ implies $\sigma_p(T) \subset [0, \infty)$. So let $\phi \in \ell^\infty(\mathbb{C})$ be given by

$$\phi(\lambda) = \begin{cases} \sqrt{\lambda} & \text{if } \lambda \in \sigma_p(T) \\ 0 & \text{otherwise} \end{cases}$$

and set $A := \phi(T)$. Since $\phi^2(\lambda) = \lambda$ on $\sigma_p(T) \cup \{0\}$, we have $A^2 = (\phi^2)(T) = T$.
Uniqueness is left as an exercise. $\square$

- We usually write $\sqrt{T} := A$ or $T^{1/2} := A$.

II.8 Unitary Equivalence for Compact Normal Operators

Def For $A \in B(H)$ and $B \in B(K)$, we say A and B are *unitarily equivalent* if there is an isomorphism $U: H \to K$ such that $UAU^{-1} = B$. In this case we write $A \cong B$.

- Note that $UAU^{-1} = B \iff UA = BU \iff A = U^{-1}BU$
- When $K = H$, $U \in B(H)$ is a unitary operator.
- For $H = \mathbb{C}^n$ and $A, B \in M_n(\mathbb{C})$, this definition is equivalent to saying A and B are similar. When A and B are self-adjoint, this holds iff they have the same eigenvalues and their corresponding eigenspaces have the same dimensions. We will see that the same result holds for compact normal operators.

Def For $T \in K(H)$, the *multiplicity function* for T is $m_T(\lambda) := \dim \ker(T-\lambda)$

- Note that $m_T(\lambda) > 0$ iff $\lambda \in \sigma_p(T)$. If $\lambda \in \sigma_p(T) \setminus \{0\}$, then $m_T(\lambda) < \infty$ by a proposition from Section II.4. However, $m_T(0)$ may be an infinite cardinal.

Thm For $T \in K(H)$ and $S \in K(K)$, $T \cong S$ iff $m_T = m_S$.

Proof ($\Rightarrow$) Let $U: H \to K$ be the isomorphism such that $UT = SU$. Observe that

$$U \ker(T-\lambda) = \ker(S-\lambda) \qquad \forall \lambda \in \mathbb{C}.$$

Indeed, $U(T-\lambda) = UT - \lambda U = SU - \lambda U = (S-\lambda)U$. Thus

$$(T-\lambda)h = 0 \Rightarrow U(T-\lambda)h = 0 \Rightarrow (S-\lambda)Uh = 0 \Rightarrow U\ker(T-\lambda) \subset \ker(S-\lambda)$$

and

$$(S-\lambda)k = 0 \Rightarrow (S-\lambda)UU^{-1}k = 0 \Rightarrow U(T-\lambda)U^{-1}k = 0 \Rightarrow U^{-1}\ker(S-\lambda) \subset \ker(T-\lambda).$$

Consequently,

$$m_S(\lambda) = \dim \ker(S-\lambda) = \dim U\ker(T-\lambda) = \dim \ker(T-\lambda) = m_T(\lambda) \quad \forall \lambda \in \mathbb{C}.$$

($\Leftarrow$) Since the multiplicity function is non-zero exactly on the point spectrum, we have $\sigma_p(T) = \sigma_p(S) =: \Lambda$. Let

$$T = \sum_{\lambda \in \Lambda} \lambda P_\lambda \qquad \text{and} \qquad S = \sum_{\lambda \in \Lambda} \lambda Q_\lambda$$

where $P_\lambda = P_{\ker(T-\lambda)}$ and $Q_\lambda = P_{\ker(S-\lambda)}$. For each $\lambda \in \Lambda$, let $U_\lambda : \ker(T-\lambda) \to \ker(S-\lambda)$ be an isomorphism, which exists since $\dim \ker(T-\lambda) = \dim \ker(S-\lambda)$. We then have

$$U_\lambda P_\lambda = Q_\lambda U_\lambda$$

Recall that $\{P_\lambda : \lambda \in \Lambda\}$ is a partition of unity on H and $\{Q_\lambda : \lambda \in \Lambda\}$ is a partition of unity on K. Consequently we can define $U: H \to K$ by $UP_\lambda := U_\lambda P_\lambda = Q_\lambda U_\lambda$. It follows that U is invertible with $U^{-1}Q_\lambda = U_\lambda^{-1} Q_\lambda = P_\lambda U_\lambda^{-1}$. Consequently

$$UTU^{-1} = \sum_{\lambda \in \Lambda} \lambda U P_\lambda U^{-1} = \sum_{\lambda \in \Lambda} \lambda Q_\lambda U_\lambda U^{-1} = S. \qquad \square$$