Distributed Capacitance Microstructure in Conductors

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Abstract. A new model for distributed capacitance in a conducting medium is introduced as a system of local RC diffusion equations coupled by a global elliptic equation. This model contains the local geometry of the distributed capacitors on which charge is stored and the exchange of current flux on their interface with the medium. The resulting degenerate initial-boundary-value problem is shown to be well posed and certain singular limits are characterized.

1. Introduction.

Remarkable progress has been made in the fabrication and understanding of novel materials that do not occur in nature. Investigators have fabricated artificial periodic superlattices, also termed layered synthetic microstructures, consisting of alternating layers of different semiconductors, different metals, or semiconductors and metals. The ability to tailor the structure of materials has led to the discovery and elucidation of many new phenomena. Multilayer structures of specific materials can be obtained directly by mechanical processing of macroscopic laminates. Such devices may also be synthesized by various techniques in which the product is formed by means of "atom by atom" processes. Such techniques include physical vapor condensation, chemical vapor deposition, and electrochemical deposition [7], [11], [5], [6], [2]. A particular example is the multilayer ceramic capacitor which is used, for example, in computer memory boards to divert spurious signals and to buffer fluctuations in the power supply. Such a capacitor is constructed in 60 to 120 layers using a ceramic such as barium titanate with a very high dielectric constant and a highly conductive metal such as a silver-palladium mixture. The ceramic layers are typically 20 to 40 micrometers thick while the metal electrode layers are usually from one to five micrometers thick [16]. Such integrated circuit and thin film technology leads to the theory of networks with distributed components. The theory of distributed networks is the natural setting in which to study the behavior of a system in which wavelengths of interest are comparable to the physical dimensions of the device [10]. With advances in technology the delay times of such a device or network become even more significant and frequently dominate the overall performance [8], [12], [15].

We shall introduce and develop here a model for a conductor in which is imbedded a continuous distribution of capacitance cells oriented with vertical normal direction. This is an idealization of, for example, a multi-layered ceramic capacitor consisting of alternating thin films of conductive and dielectric horizontal layers, as well as of a variety of materials fabricated by intercalation techniques. Among the difficulties in constructing the model are that the physical situation contains such a large number of individual cells or layers of dielectric material and that these are of such a small size compared to the containing conductor that the resulting "real" problem is extremely singular. It will certainly not be described adequately by classical approaches in which the fine scale structure is lost in the averaging process. The shape and configuration of these cells may influence the response of the system, so this information should also be included in the model, at least until its effect is understood and perhaps quantified. For the special case of a purely layered material, the Layered Medium Equation was introduced in [3]. This was shown to arise as the limit by homogenization of the classical but singular case of discretely layered systems. Here we intend to include also the more general case of a distribution of flat cells of arbitrary shape and to show these two models are consistent when the cells are uniformly oriented.

The approach we follow here is to develop a two-scale model. This takes the form of a continuum of diffusion equations, each of which describes the conduction and storage of charge on the micro-scale of an individual cell at a specific point in the global medium, and a single elliptic equation which specifies the interconnection by conservation of charge on the macro-scale of the global medium. An important aspect of this model is that it contains the fine scale geometry of the individual capacitors as well as the current flux across the intricate interface by which they are connected to the global current field. In Section 2 we derive this microstructure model of distributed capacitance, and formulate it in a variational structure. (Such a model could certainly be obtained by homogenization of a discrete system, but we have not addressed that question here.) The resulting initial-boundary value problem is shown to be well posed in Section 3, and there we discuss the regularity properties of the solution. In Section 4 we show that previously studied models such as the Layered Medium Equation are obtained as singular limits of our micro-structure model as the distributed capacitors are approximated by single points of charge storage. (Of course the geometry of the distributed capacitors is lost in this limit.) The dependence on other parameters and corresponding singular limits will also be presented.

2. Distributed Capacitance Model.

Let Ω be a bounded domain in $\mathbb{R}^3$. For each point $x \in \Omega$ let there be given a bounded cylindrical domain Ω_x in $\mathbb{R}^3$ of the form $\Omega_x = S_x \times [-\frac{h}{2}, \frac{h}{2}]$. Here S_x is the cross section in $\mathbb{R}^2$ and $h > 0$ is the thickness. Each such Ω_x represents a capacitor at the point $x \in \Omega$. Denote the points of Ω_x by $\bar{y} = (y, y_3)$ with $y = (y_1, y_2) \in S_x$ and $|y_3| \leq h/2$. The primary variables in our model are the voltage distribution $u(x,t)$ in the global region and the locally distributed voltage *difference* $U(x,y,t)$ across the capacitor Ω_x at $y \in S_x$. Assume the vertical component of the global voltage gradient induces a current into the (horizontally oriented) capacitor at $x \in \Omega$ of magnitude $G(U(x,y,t) - hu_3(x,t))$, $y \in S_x$, where u_3 denotes the derivative in the x_3 (vertical) direction. That is, the capacitor is connected to the surrounding global voltage field through a surface-distributed conductance of magnitude G. If the distributed capacitance and conductance of the capacitor are denoted by C and K, respectively, and if any distributed current sources for the capacitor are denoted by $F(x,y,t)$, then the conservation of charge leads to the classical RC diffusion equation at each point $x \in \Omega$ given by

$$CU_t - \vec{\nabla}_y \cdot K\vec{\nabla}_y U + G(U - hu_3) = F \ , \qquad y \in S_x \ . \tag{2.1.a}$$

Thus the vertical component of the voltage gradient drives the capacitor as a distributed source, Ghu_3. Similarly this voltage drop induces across the boundary of the capacitor in the horizontal normal direction, ν, a current of magnitude $hg(U(x,y,t) - hu_3(x,t))$, where the surface-distributed conductance g on the boundary $\partial S_x \times [-\frac{h}{2}, \frac{h}{2}]$ gives the effective line-distributed conductance hg on ∂S_x. To see this, note that on ∂S_x we have, respectively,

$$K\frac{\partial U^{\pm}}{\partial \nu} + hg\big(U^{\pm} - (u \pm \frac{h}{2}u_3)\big) = 0$$

in terms of the absolute voltages, $U^{\pm}$, at $y_3 = \pm\frac{h}{2}$. Since $U = U^{+} - U^{-}$, this leads to the boundary condition

(2.1.b) $$K\frac{\partial U}{\partial \nu} + hg(U - hu_3) = 0 \ , \qquad y \in \partial S_x \ .$$

Finally, the global current flux arises in two parts, one from the distributed voltage, $u(x,t)$, and the second from the vertical current exiting the capacitor at x, and this is given by

$$\vec{\jmath} = -\left\{ k\vec{\nabla}u + \frac{\vec{e}_3}{|S_x|}\left(\int_{\partial S_x} K\frac{\partial U}{\partial \nu}\, ds + \int_{S_x} G(hu_3 - U)\, dy \right) \right\} \ .$$

Thus the conservation of charge on the global scale requires that

(2.1.c) $$\vec{\nabla} \cdot \vec{\jmath} = f \ , \qquad x \in \Omega \ ,$$

where $f(x,t)$ denotes any charge sources distributed over Ω. The model for conduction in this medium is completed by a condition for current flow on the boundary, $\partial\Omega$, for example, in the case of a grounded boundary,

(2.1.d) $$u(x,t) = 0 \ , \qquad x \in \partial\Omega \ , \ t > 0$$

and initial values for the charge distributions,

(2.2) $$CU(x,y,0) = CU_0(x,y) \ , \qquad x \in \Omega \ , \ y \in S_x \ .$$

The system (2.1) is our distributed RC network model for distributed capacitance. It consists of the global field equation (2.1.c) which couples the distribution of capacitors, each of which is coupled to the vertical current field by (2.1.a) and to the horizontal current field by (2.1.b). Note that the total charge rate of the capacitor Ω_x is given by

$$\frac{d}{dt}\int_{S_x} CU\, dy = \int_{S_x} \big[F + G(hu_3 - U)\big]\, dy + \int_{\partial S_x} hg(hu_3 - U)\, ds$$

$$= \left\{ \int_{S_x} F\, dy + \int_{\partial \Omega_x} J\, dS \right\}$$

in which the function

$$J = \begin{cases} G(hu_3 - U)/2 & \text{at} \quad \bar{y} = (y, \pm\frac{h}{2}),\ y \in S_x \\ g(hu_3 - U) & \text{at} \quad \bar{y} = (y, y_3),\ y \in \partial S_x \end{cases}$$

is the current flux across the boundary $\partial\Omega_x$ on the top, bottom, (S_x), and sides, (∂S_x), respectively. Thus we have a system of partial differential equations of mixed parabolic-elliptic type that are coupled through interfaces of both internal and boundary type.

Next we develop the variational statement of the system (2.1), a weak form of the problem formulated in Hilbert spaces of Sobolev type. Denote by $L^2(\Omega)$ the space of (equivalence classes of) Lebesgue square-integrable functions on Ω, and let $C_0^\infty(\Omega)$ denote the subspace of infinitely differentiable functions with compact support. $H^1(\Omega)$ is the Hilbert space of functions in $L^2(\Omega)$ for which each partial derivative belongs to $L^2(\Omega)$. We shall let $H_0^1(\Omega)$ be the subspace obtained as the closure in $H^1(\Omega)$ of $C_0^\infty(\Omega)$. See [1] for information on these Sobolev spaces. In order to prescribe a measurable family of cells, $\{S_x, x \in \Omega\}$, let $Q \subset \Omega \times \mathbb{R}^2$ be a given measurable set, and set $S_x = \{y \in \mathbb{R}^2 : (x, y) \in Q\}$. Each S_x is measurable in $\mathbb{R}^2$ and by zero-extension we identify $L^2(Q) \hookrightarrow L^2(\Omega \times \mathbb{R}^2)$ and each $L^2(S_x) \hookrightarrow L^2(\mathbb{R}^2)$. Thus we obtain the identification

$$L^2(Q) \cong \Big\{U \in L^2\big(\Omega, L^2(\mathbb{R}^2)\big) : U(x) \in L^2(S_x)\ , \quad \text{a.e.} \quad x \in \Omega\Big\}\ .$$

Hereafter we shall denote this Hilbert space with scalar-product

$$(U, \Phi)_{\mathcal{H}} = \int_\Omega \left\{ \frac{1}{|\Omega_x|} \int_{S_x} U(x, y)\Phi(x, y)\, dy \right\} dx$$

by $\mathcal{H} = L^2(\Omega, L^2(S_x))$, and we shall set $H_x = L^2(S_x)$ for each $x \in \Omega$. The *state space* for our problems will be the product

$$H \equiv L^2(\Omega) \times \mathcal{H} \cong L^2(\Omega) \times L^2(Q)\ .$$

Suppose $\{W_x : x \in \Omega\}$ is the collection of Sobolev spaces $W_x = H^1(S_x)$ so that each W_x is continuously imbedded in H_x, uniformly for $x \in \Omega$. It follows that the direct sum

$$\mathcal{W} \equiv L^2(\Omega, W_x) \equiv \Big\{U \in \mathcal{H} : U(x) \in W_x\ , \quad \text{a.e.} \quad x \in \Omega\ , \quad \text{and} \quad \int_\Omega \|U(x)\|^2_{W_x}\, dx < \infty\Big\}$$

is a Hilbert space; the product

$$V \equiv H_0^1(\Omega) \times \mathcal{W}$$

will be the *energy space* for our system (2.1). We shall use a variety of such subspaces of $\mathcal{H}$ which can be constructed in this manner. Moreover we shall assume that each S_x is a bounded domain in $\mathbb{R}^2$ which lies locally on one side of its boundary, ∂S_x, and ∂S_x is a smooth curve in $\mathbb{R}^2$. We also assume the trace maps $\gamma_x : W_x \to L^2(\partial S_x)$ are *uniformly bounded*, so that we may define the *distributed trace* $\gamma(U) \in L^2(\Omega, L^2(\partial S_x))$ by $\gamma(U)(x,s) = \gamma_x U(s)$, $s \in \partial S_x$, $x \in \Omega$, and so γ is bounded and linear from $\mathcal{W}$ into $L^2(\Omega, L^2(\partial S_x))$.

In order to get the weak formulation of (2.1), suppose u, U is an appropriately smooth solution of (2.1), multiply (2.1.a) by a corresponding $\Phi(x,y)$, integrate over S_x and use (2.1.b) to obtain

$$\text{(2.3.a)} \qquad \frac{1}{|\Omega_x|}\int_{S_x}\left\{CU_t\Phi + K\vec{\nabla}_y U\cdot\vec{\nabla}_y\Phi + G(U - hu_3)\Phi\right\}dy$$
$$+\frac{1}{|S_x|}\int_{\partial S_x} g(\gamma U - hu_3)\gamma\Phi\,ds = \frac{1}{|\Omega_x|}\int_{S_x}F\Phi\,dy\ .$$

Here we have used $|\Omega_x| = h|S_x|$. Similarly multiply (2.1.c) by $\varphi \in H_0^1(\Omega)$ and integrate over Ω to obtain

$$\text{(2.3.b)} \qquad \int_\Omega\left\{k\vec{\nabla}u\cdot\vec{\nabla}\varphi + \frac{1}{|S_x|}\left(\int_{\partial S_x}K\frac{\partial U}{\partial\nu}\,ds + \int_{S_x}G(hu_3 - U)\,dy\right)\varphi_3\right\}dx$$
$$= \int_\Omega f\varphi\,dx\ .$$

Finally, we use (2.1.b) above and add (2.3.b) to the integral of (2.3.a) over Ω to obtain

$$\text{(2.3)} \quad \int_\Omega\left\{k\vec{\nabla}u\cdot\vec{\nabla}\varphi + \frac{1}{|\Omega_x|}\int_{S_x}\left[cU_t\Phi + K\vec{\nabla}_y U\cdot\vec{\nabla}_y\Phi + G(hu_3 - U)(h\varphi_3 - \Phi)\right]dy\right.$$
$$\left.+\frac{1}{|S_x|}\int_{\partial S_x}g(hu_3 - \gamma U)(h\varphi_3 - \gamma\Phi)\,ds\right\}dx$$
$$= \int_\Omega\left\{f\varphi + \frac{1}{|\Omega_x|}\int_{S_x}F\Phi\,dy\right\}dx\ .$$

We define a *generalized solution* of the system (2.1) to be a pair of functions

$$u \in L^2\big(0,T;H_0^1(\Omega)\big) \quad , \quad U \in L^2(0,T;\mathcal{W})$$

with $U_t \in L^2(0,T;\mathcal{W}')$ for which (2.3) holds for almost every $t>0$ and every pair $\varphi \in H_0^1(\Omega)$, $\Phi \in \mathcal{W}$. Note that we have the continuous imbeddings $\mathcal{W} \subset \mathcal{H} \subset \mathcal{W}'$. Since $U \in L^2(0,T;\mathcal{W})$ and the derivative $U_t \in L^2(0,T;\mathcal{W}')$, it follows (eg., from p.176 of [4]) that after modification on a null set we have $U \in C(0,T;\mathcal{H})$. Thus, the initial condition (2.2) is meaningful, and

$$(2.4) \qquad |U(t)|_{\mathcal{H}}^2 \le |U(0)|_{\mathcal{H}}^2 + 2\int_0^t \big(U'(s),U(s)\big)_{\mathcal{H}}\,ds\ , \qquad 0 \le t \le T\ .$$

3. The Cauchy Problem.

We shall show that the Cauchy problem for the system (2.1) constitutes a well-posed problem in its generalized form given by (2.3). Moreover we shall show that in the case of "$f=0$" the dynamics is governed by a *holomorphic semigroup* in the Hilbert space $\mathcal{H}$.

The variational form of the problem (2.1) is given by (2.3). This can be expressed efficiently in terms of the pair of continuous linear operators from the product spaces H and V into their respective duals, H' and V', given by

$$(3.1.a) \qquad \ell\tilde{u}(\tilde{\varphi}) = \int_\Omega \frac{1}{|\Omega_x|}\int_{S_x} CU\Phi\,dy\,dx\ , \qquad \tilde{u}=[u,U]\ ,\ \tilde{\varphi}=[\varphi,\Phi]\in H\ ,$$

$$(3.1.b) \qquad \mathcal{L}\tilde{u}(\tilde{\varphi}) = \int_\Omega \Big\{ k\vec{\nabla}u\cdot\vec{\nabla}\varphi + \frac{1}{|\Omega_x|}\int_{S_x}\big(K\vec{\nabla}_y U\cdot\vec{\nabla}_y\Phi + G(hu_3-U)(h\varphi_3-\Phi)\big)\,dy$$
$$+\frac{1}{|S_x|}\int_{\partial S_x} g(hu_3-\gamma U)(h\varphi_3-\gamma\Phi)\,ds\Big\}\,dx\ , \qquad \tilde{u},\tilde{\varphi}\in V\ .$$

Here we shall assume the coefficient functions C, k, K are given positive constants, and that g, G are non-negative constants, although they could as well be corresponding bounded functions. A generalized solution of (2.1) is then a pair $\tilde{u}=[u,U]\in L^2(0,T;V)$ for which

$$(3.2) \qquad \frac{d}{dt}\ell\tilde{u}(t) + \mathcal{L}\tilde{u}(t) = \tilde{f}(t) \ \text{ in } \ V'\ , \quad \text{a.e.} \quad t>0\ .$$

Here we have set $\tilde{f}=[f,F]$. That (2.1) is well-posed follows directly from the results of [13]. Specifically, we obtain the following

Theorem 1. *Given $\tilde{f} = [f, F] \in L^2(0,T;V')$ and $U_0 \in \mathcal{H}$, there exists a unique generalized solution $\tilde{u}$ of* (3.2) *and* (2.2), *and it satisfies*

$$\|U\|_{C([0,T],\mathcal{H})} + \|\tilde{u}\|_{L^2(0,T;V)} \leq \text{const.}\left(\|\tilde{f}\|_{L^2(0,T;V')} + \|U_0\|_{\mathcal{H}}\right) . \tag{3.3}$$

Proof outline. We have continuous and symmetric linear operators ℓ, $\mathcal{L}$ on the Hilbert spaces $V \subset H$, respectively, and the coercive-estimate

$$(\ell + \mathcal{L})\tilde{u}(\tilde{u}) \geq c_0 \|\tilde{u}\|_V^2 , \qquad \tilde{u} \in V \tag{3.4}$$

holds for some $c_0 > 0$. It follows directly from Section 2 of [13] and (2.4) that there exists a unique solution of (3.2) and (2.2), and this solution satisfies (3.3). ∎

The preceding provides a generalized notion of solution of our problem under very general conditions on the data. Clearly similar results on the well-posedness of the problem in this setting hold also in much more general situations, e.g., of time-dependent operators and spaces as indicated in [13]. We shall next show that the solution is very smooth when $f = 0$ and, moreover, that the dynamics is governed by a holomorphic semigroup on $\mathcal{H}$ with the corresponding regularizing effects of a *parabolic* system. To this end, we begin with the operators $\ell : H \to H'$ and $\mathcal{L} : V \to V'$ given above and define the domain

$$D(A) \equiv \left\{U \in \mathcal{W} : \tilde{u} = [u, U] \in V \text{ for some } u \in H_0^1(\Omega) \text{ with } \mathcal{L}\tilde{u} \in \{0\} \times \mathcal{H}\right\} .$$

Then the operator $A : D(A) \to \mathcal{H}$ is given by $A(U) = F \in \mathcal{H}$ if $U \in D(A)$ and $\tilde{u} = [u, U]$ as above satisfies

$$\mathcal{L}\tilde{u}(\tilde{\varphi}) = \ell\big([0, F]\big)(\tilde{\varphi}) , \qquad \tilde{\varphi} = [\varphi, \Phi] \in V .$$

Since $\mathcal{L}\tilde{u}(\tilde{u}) \geq c_1 \|u\|_{H_0^1}^2$ for some $c_1 > 0$, such a $\tilde{u}$ is unique.

Lemma. *The operator A is symmetric and m-accretive in $\mathcal{H}$.*

Proof. With $U \in D(A)$ and $\tilde{u}$ as above we have

$$(CAU, U)_{\mathcal{H}} = \ell\big([0, AU]\big)(\tilde{u}) = \mathcal{L}\tilde{u}(\tilde{u}) \geq 0$$

so A is accretive. Furthermore, since ℓ and $\mathcal{L}$ are symmetric, we have for any $\Phi \in D(A)$ and corresponding $\tilde{\varphi} = [\varphi, \Phi] \in V$

$$(CAU, \Phi)_{\mathcal{H}} = \ell\big([0, AU]\big)(\tilde{\varphi}) = \mathcal{L}\tilde{u}(\tilde{\varphi}) = \mathcal{L}\tilde{\varphi}(\tilde{u}) = (cU, A\Phi)_{\mathcal{H}} ,$$

so A is symmetric. Finally, note that for any $\alpha > 0$ we have

$$(\alpha\ell + \mathcal{L})\tilde{u}(\tilde{\varphi}) = \big(C(\alpha I + A)U, \Phi\big)_{\mathcal{H}} , \qquad U \in D(A) , \ \tilde{\varphi} \in V ,$$

so the operator $\alpha I + A$ arises from the corresponding $\alpha\ell + \mathcal{L} : V \to V'$. Since this is V-coercive for each $\alpha > 0$, it follows from standard techniques, e.g., [14], that $\alpha I + A$ is onto $\mathcal{H}$, hence, A is m-accretive. ∎

¿From the Lemma it follows that $-A$ is the generator of a holomorphic semigroup [9] on $\mathcal{H}$. This gives the following result.

Theorem 2. *Assume the sets $Q = \Pi\{S_x : x \in \Omega\}$ are given as in Section 2 with uniformly bounded trace maps and smooth boundaries, ∂S_x. Let the strictly-positive constants C, k, K, h and non-negative constants g, G be given. For each $U_0 \in L^2(Q)$ and Hölder continuous $F : [0,T] \to L^2(Q)$ there exists a unique pair $u \in C((0,T], H_0^1(\Omega))$, $U \in C([0,T], L^2(Q))$ with $U \in C^1((0,T], L^2(Q))$ which is a generalized solution of* (2.1), (2.2). *In particular, they satisfy for each $t > 0$*

$$\text{(3.5.a)} \quad \int_\Omega \Big\{ k\vec{\nabla}u(t) \cdot \vec{\nabla}\varphi + \Big(\frac{1}{|\Omega_x|} \int_{S_x} G\big(hu_3(t) - U(t)\big)\, dy$$

$$+ \frac{1}{|S_x|} \int_{\partial S_x} g\big(hu_3(t) - \gamma U(t)\big)\, ds \Big) h\varphi_3 \, dx = 0 , \qquad \varphi \in H_0^1(\Omega) ,$$

$$\text{(3.5.b)} \quad \int_{S_x} CU'(t)\Phi + K\vec{\nabla}_y U(t) \cdot \vec{\nabla}_y \Phi + G\big(U(t) - hu_3(t)\big)\Phi \, dy$$

$$+ \int_{\partial S_x} hg\big(\gamma U(t) - hu_3(t)\big)\gamma\Phi \, ds = \int_{S_x} F(t)\Phi \, dy , \quad \Phi \in L^2\big(\Omega, H^1(S_x)\big) .$$

The essential gain in regularity of Theorem 2 over Theorem 1 is that the derivative U_t is now in $C((0,T]; L^2(Q))$. Thus, at almost every $x \in \Omega$, each term in the *strong* form (2.1.a) of the equation in (3.5.b) is in $L^2(S_x)$. When the data in this parabolic problem is appropriately smooth, we obtain a corresponding regularity result: $U(x,\cdot,t) \in H^2(S_x)$ at a.e. $x \in \Omega$ for each $t \in (0,T]$. On the other hand, the coefficient of $\vec{e}_3$ in the flux $\vec{\jmath}$ is at best in $L^2(\Omega)$ because of U, so (2.1.c) holds only in the *weak* form (3.5.a), i.e., in $H_0^1(\Omega)'$, so there is no corresponding regularity result for the first component. This is, of course, fully expected, since the dynamics of this *degenerate parabolic* system involves only the second component, U. Finally, we also note that the initial data in Theorem 2 is chosen from $\mathcal{H} = L^2(Q)$ whereas for the non-holomorphic case one would need to require $U_0 \in D(A)$.

4. Limiting Cases.

Our objective here is to find the form of the problem that results from letting any one (or more) of the parameters g, G, K increase without bound. We shall show that letting $g \to \infty$ merely leads to a problem similar to (2.1) but with (2.1.b) replaced by a *Dirichlet condition*, the limiting case as $G \to \infty$ is the *layered medium equation* [3], and the limit as $K \to \infty$ gives the *regularized layered medium equation*. We shall treat these various cases in a unified way as follows. First, multiply the relevant coefficient by $1/\varepsilon$ and consider the corresponding problem with generalized solution $\tilde{u}_\varepsilon = [u_\varepsilon, U_\varepsilon]$. Then show $\tilde{u}_\varepsilon \to \tilde{u}_0$ in the appropriate sense where $\tilde{u}_0$ is the generalized solution of the limiting problem. Finally, we shall characterize the limiting problem for each of these three cases.

The convergence results will each follow from the following.

Theorem 3. (a) *Let the spaces* $V, H, \mathcal{H}$, *functions* $\tilde{f} \in L^2(0,T;H)$, $U_\varepsilon^0 \in \mathcal{H}$, $0 < \varepsilon < 1$, *and operators* $\ell, \mathcal{L}$ *be given as in Theorem 1. Thus setting* $\mathcal{V} \equiv L^2(0,T;V)$ *and* $B \equiv C([0,T],\mathcal{H})$ *we have for some* $c_1 > 0$ *from (2.4) and (3.4)*

$$\text{(4.1)} \qquad \int_0^T \left(\frac{d}{dt}\ell + \mathcal{L}\right)\tilde{u}(\tilde{u})\,dt + |U(0)|_{\mathcal{H}}^2 \geq c_1\left(\|\tilde{u}\|_{\mathcal{V}}^2 + |U(T)|^2\right) , \qquad \tilde{u} = [u,U] \in \mathcal{V} .$$

Let $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ *with each part being continuous, linear, symmetric, and non-negative; define* $\mathcal{L}_\varepsilon \equiv \mathcal{L}_0 + \frac{1}{\varepsilon}\mathcal{L}_1$, $0 < \varepsilon < 1$. *Then for each* ε, $0 < \varepsilon < 1$, *there is a unique generalized solution* $\tilde{u}_\varepsilon = [u_\varepsilon, U_\varepsilon]$ *of*

$$\text{(4.2)} \qquad \tilde{u}_\varepsilon \in \mathcal{V} : \frac{d}{dt}\ell\tilde{u}_\varepsilon + \mathcal{L}_\varepsilon\tilde{u}_\varepsilon = \tilde{f} \;\; \text{in} \;\; \mathcal{V}'$$

with $U_\varepsilon(0) = U_\varepsilon^0$ *in* $\mathcal{H}$.

(b) *Let* $V_0 \equiv \ker(\mathcal{L}_1) = \{\tilde{v} \in V : \mathcal{L}_1\tilde{v} = 0\}$, *denote the closure in* $\mathcal{H}$ *of* $\{U : [u,U] \in V_0\}$ *by* $\mathcal{H}_0$, *and let* $H_0 \equiv L^2(\Omega) \times \mathcal{H}_0$. *Set* $\mathcal{V}_0 \equiv L^2(0,T;V_0)$, $B_0 = C([0,T],\mathcal{H}_0)$, *and define the restriction* $\tilde{f}_0 \equiv \tilde{f}|_{V_0}$. *Let* $U_0^0 \in \mathcal{H}_0$. *Then there is a unique generalized solution* $\tilde{u}_0 = [u_0, U_0]$ *of*

$$\tilde{u}_0 \in \mathcal{V}_0 : \frac{d}{dt}\ell\tilde{u}_0 + \mathcal{L}_0\tilde{u}_0 = \tilde{f}_0 \ \ in \ \ \mathcal{V}_0' \tag{4.3}$$

with $U_0(0) = U_0^0$ *in* $\mathcal{H}_0$.

(c) *If* $U_\varepsilon^0 \to U_0^0$ *in* $\mathcal{H}$ *as* $\varepsilon \to 0$, *then we have the convergence* $\tilde{u}_\varepsilon \to \tilde{u}_0$ *in* $\mathcal{V}$ *and* $U_\varepsilon \to U_0$ *in* B.

Proof. Part (a) follows directly from Theorem 1, and (b) is likewise immediately obtained by replacing $V, H, \mathcal{H}$ by the corresponding subspaces, $V_0, H_0, \mathcal{H}_0$, respectively. To prove (c), let $\tilde{u}_\varepsilon, \tilde{u}_0$ be the indicated solutions of (4.2) and (4.3). Applying (4.1) to the difference $\tilde{u} = \tilde{u}_\varepsilon - \tilde{u}_0$ gives the estimate

$$\begin{aligned} c_1\Big(\|\tilde{u}_\varepsilon - \tilde{u}_0\|_{\mathcal{V}}^2 + |U_\varepsilon(T) - U_0(T)|^2\Big) &\le \int_0^T \Big(\frac{d}{dt}\ell + \mathcal{L}_\varepsilon\Big)(\tilde{u}_\varepsilon - \tilde{u}_0)(\tilde{u}_\varepsilon - \tilde{u}_0)\,dt + |U_\varepsilon^0 - U_0^0|_{\mathcal{H}}^2 \\ &= \Big\langle \tilde{f} - \Big(\frac{d}{dt}\ell + \mathcal{L}_0\Big)\tilde{u}_0, \tilde{u}_\varepsilon - \tilde{u}_0\Big\rangle + |U_\varepsilon^0 - U_0^0|_{\mathcal{H}}^2 \end{aligned} \tag{4.4}$$

since $\mathcal{L}_1\tilde{u}_0 = 0$. Thus (by passing to a subsequence) we obtain a $\tilde{u}_1 \in \mathcal{V}$ for which $\tilde{u}_\varepsilon \rightharpoonup \tilde{u}_1$ (weakly). From (4.1) and (4.2) we find that $\{\frac{1}{\varepsilon}\mathcal{L}_1\tilde{u}_\varepsilon(\tilde{u}_\varepsilon)\}$ is bounded, so by weak-lower-semicontinuity there follows

$$\mathcal{L}_1\tilde{u}_1(\tilde{u}_1) \le \liminf_{\varepsilon\to 0} \mathcal{L}_1\tilde{u}_\varepsilon(\tilde{u}_\varepsilon) = 0 \ ,$$

hence $\tilde{u}_1 \in \mathcal{V}_0$. Furthermore we have $(\frac{d}{dt}\ell + \mathcal{L}_0)\tilde{u}_\varepsilon \rightharpoonup (\frac{d}{dt}\ell + \mathcal{L}_0)\tilde{u}_1 = \tilde{f}_0$ in $\mathcal{V}_0'$. Applying (4.1) to $\tilde{u}_\varepsilon$, the solutions of (4.2), shows that $\{U_\varepsilon\}$ is bounded in B and so (by passing to a further subsequence) we have $U_\varepsilon(0) \rightharpoonup U_1(0)$ in $\mathcal{H}_0$, hence $U_1(0) = U_0^0$. But by uniqueness we have then $\tilde{u}_1 = \tilde{u}_0$, and then from (4.4) we get $\tilde{u}_\varepsilon \to \tilde{u}_0$ in $\mathcal{V}$ and $U_\varepsilon \to U_0$ in B as desired.

We shall apply Theorem 3 to characterize the limiting form of our problem (2.1) as the coefficients g, G or K increase without bound. For the first case we set (cf. (3.1.b))

$$\mathcal{L}_1\tilde{u}(\tilde{\varphi}) = \int_\Omega \frac{1}{|S_x|}\int_{\partial S_x} g(hu_3 - \gamma U)(h\varphi_3 - \gamma\Phi)\,ds\,dx \ , \qquad \tilde{u}, \tilde{\varphi} \in V$$

and then let $\mathcal{L}_0 \equiv \mathcal{L} - \mathcal{L}_1$. Thus (4.2) is the generalized form of problem (2.1) with g replaced by $\frac{1}{\varepsilon}g$, and according to Theorem 3 its solution converges as $\varepsilon \to 0$ to the solution of (4.3). In this case we have $V_0 = V_g$, where

$$V_g \equiv \Big\{\tilde{u} = [u, U] \in V : \gamma U = h u_3 \;\text{ in }\; L^2\big(\Omega, L^2(\partial S_x)\big)\Big\} ,$$

and from calculations as in Section 2 we find that (4.3) is the weak formulation of (2.1) with (2.1.b) replaced by the *Dirichlet condition*

$$\text{(2.1.b}')\qquad U(x,y) = h u_3(x) , \qquad y \in \partial S_x , \; x \in \Omega .$$

This limiting problem (2.1′) is the *matched model* in which the distributed voltage differences, γU, on the capacitor boundaries are in perfect contact with the global voltage vertical gradient, hu_3. Theorem 3 gives a quantitative statement of the dependence of this model on a finite conductance between the boundary differences and the vertical gradient of voltage.

Next we consider the case of increasing G. For this we define

$$\mathcal{L}_1 \tilde{u}(\tilde{\varphi}) = \int_\Omega \frac{1}{|\Omega_x|} \int_{S_x} G(hu_3 - U)(h\varphi_3 - \Phi)\, dy\, dx , \qquad \tilde{u}, \tilde{\varphi} \in V ,$$

and let $\mathcal{L}_0 = \mathcal{L} - \mathcal{L}_1$, where $\mathcal{L}$ is given as before by (3.1.b). Thus (4.2) corresponds to (2.1) with G replaced by $\frac{1}{\varepsilon}G$. Here we have $V_0 = V_G$ with

$$V_G \equiv \Big\{\tilde{u} = [u, U] \in V : U = h u_3 \;\text{ in }\; L^2\big(\Omega, L^2(S_x)\big)\Big\} ,$$

hence, $V_G \cong H_0^1(\Omega)$. From (2.3) we find that (4.3) gives a generalized solution of

$$\text{(4.5)}\qquad \begin{aligned} & u \in L^2\big(0,T; H_0^1(\Omega)\big) \quad , \quad u_3 \in H^1\big(0,T; L^2(\Omega)\big) \\ & -\vec{\nabla} \cdot \left\{ k\vec{\nabla} u + \frac{\vec{e}_3}{|S_x|} \int_{S_x} C\, dy \cdot \frac{\partial}{\partial t}(hu_3) \right\} = f - \frac{\partial}{\partial x_3} \frac{1}{|S_x|} \int_{S_x} F(\cdot, y)\, dy . \end{aligned}$$

This is the *layered medium equation* [3], and Theorem 3 shows that it is obtained in the limit from (2.1) by letting G (more precisely, $\frac{1}{\varepsilon}G$) $\to \infty$.

Finally, we consider the case of $K \to \infty$. For this we define

$$\mathcal{L}_1 \tilde{u}(\tilde{\varphi}) = \int_\Omega \frac{1}{|\Omega_x|} \int_{S_x} (K-1)\vec{\nabla}_y U \cdot \vec{\nabla}_y \Phi\, dy\, dx , \qquad \tilde{u}, \tilde{\varphi} \in V$$

and $\mathcal{L}_0 = \mathcal{L} - \mathcal{L}_1$ as before. The kernel of $\mathcal{L}_1$ consists of those members of $\mathcal{W}$ which are independent of y, so $V_0 = V_K$,

$$V_K \equiv \big\{\tilde{u} = [u, U] \in V : U(x, y) = v(x) \ , \quad \text{a.e.} \quad (x, y) \in Q\big\} \ ,$$

hence, $V_K \cong H_0^1(\Omega) \times L^2(\Omega)$. From (2.3) we find that (4.3) is the weak formulation of the system

$$u \in L^2\big(0, T; H_0^1(\Omega)\big) \quad , \quad v \in H^1\big(0, T; L^2(\Omega)\big) \ ,$$

$$-\vec{\nabla} \cdot \left\{ k\vec{\nabla} u + \frac{\vec{e}_3}{|S_x|} \left(\int_{S_x} G\, dy + \int_{\partial S_x} hg\, ds \right) (hu_3 - v) \right\} = f \ , \tag{4.6.a}$$

$$\left(\frac{1}{|S_x|} \int_{S_x} C\, dy \right) \frac{\partial v}{\partial t} + \frac{1}{|S_x|} \left(\int_{S_x} G\, dy + \int_{\partial S_x} hg\, ds \right) (v - hu_3) = \frac{1}{|S_x|} \int_{S_x} F(\cdot, y)\, dy \ . \tag{4.6.b}$$

According to Theorem 3, the solution of (2.1) converges to that of (4.6) as $K \to \infty$. By eliminating v from this system in the *uniform* case of $S_x = S$, $x \in \Omega$, we obtain the *regularized layered medium equation* [12]

$$C \frac{\partial}{\partial t} \left(\partial_3 h u_3 + \frac{1}{\tilde{g}} \vec{\nabla} \cdot k \vec{\nabla} u + \frac{1}{\tilde{g}} f \right) + \vec{\nabla} \cdot k \vec{\nabla} u = -f + \partial_3 \left(\frac{1}{|S|} \int_S F\, dy \right) \tag{4.7}$$

in which $\tilde{g} \equiv \frac{1}{|S|}(\int_S G\, dy + \int_{\partial S} hg\, ds)$ is the total conductance between each capacitor and the enclosing conducting field. It follows by similar arguments that the solution of (4.7) converges to that of (4.5) as $\tilde{g} \to \infty$; this convergence and its consequences for the propogation of singularities in initial data are studied in [12].

References

[1] R.A. Adams, "Sobolev Spaces", Academic Press, New York, 1975.

[2] T.W. Barbee, Jr., *Layered synthetic microstructures: reflecting media for x-ray optic elements and diffracting structures for the study of condensed matter*, Superlattices and Microstructures **1** (1985), 311–326.

[3] M.-P. Bosse and R.E. Showalter, *Homogenization of the layered medium equation*, Applicable Analysis, **32** (1989), 183–202.

[4] R.W. Carroll, "Abstract Methods in Partial Differential Equations", Harper and Row, New York, 1969.

[5] G. Choe, A.P. Valanju, R.W. Walser, *Structural inhomogeniety and magnetic properties of amorphous $Gd_{17}Co_{83}$ compositionally modulated thin films*, J. Appl. Phys. **67** (1990), 5674–5676.

[6] G.H. Doeher, *The potential of n-i-p-i doping superlattices for novel semiconductor devices*, Superlattices and Microstructures **1** (1985), 279–287.

[7] M.S. Dresselhaus, *Modifying materials by intercalation*, Physics Today **37** (March 1984), 60–68.

[8] N. Gopal, *Fast evaluation of VLSI interconnect structures using moment-matching methods*, Dissertation, The University of Texas at Austin, December 1992.

[9] T. Kato, "Perturbation Theory for Linear Operators," Springer, 1966.

[10] J.J. Kelly and M.S. Ghausi, "Introduction to Distributed Parametric Networks," Holt, Rinehart and Winston, New York, 1968.

[11] M.M.Z. Kharadly, W. Jackson, *The properties of artificial dielectrics comprising arrays of conducting elements*, Proc. I.E.E.E. (London) **100** (1953), 199–212.

[12] L. Packer and R.E. Showalter, *The Regularized Layered Medium Equation*, in preparation.

[13] R.E. Showalter, *Degenerate evolution equations*, Indiana Univ. Math. J., **23** (1974), 655–677.

[14] R.E. Showalter, "Hilbert Space Method for Partial Differential Equations", Pitman, 1977.

[15] R.E. Showalter and C. Snyder, *A distributed RC network model with dielectric loss*, IEEE Transactions on Circuits and Systems, **33** (1986), 707–710.

[16] D.M. Trotter, *Capacitors*, Scientific American, July **259** (1988), 86–90B.