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MONOGENESIS OF THE RINGS OF INTEGERS IN CERTAIN IMAGINARY ABELIAN FIELDS

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Abstract. In this paper we consider a subfield K in a cyclotomic field k_m of conductor m such that $[k_m : K] = 2$ in the cases of $m = \ell p^n$ with a prime p, where $\ell = 4$ or $p > \ell = 3$. Then the theme is to know whether the ring of integers in K has a power basis or does not.

§1. Introduction

Let F be an algebraic number field over the rationals $\boldsymbol{Q}$. We denote the ring of integers in F by $\boldsymbol{Z}_F$. If we have $\boldsymbol{Z}_F = \boldsymbol{Z}[\alpha]$ for an element α of $\boldsymbol{Z}_F$, then it is said that α generates a power basis of the ring $\boldsymbol{Z}_F$ or simply $\boldsymbol{Z}_F$ has a power basis. The ring $\boldsymbol{Z}_F$ is called monogenic if $\boldsymbol{Z}_F$ has a power basis, otherwise $\boldsymbol{Z}_F$ is said to be non-monogenic. To determine whether the ring of integers in a field is monogenic or not is proposed as an unsolved problem in [Nar]. This problem is treated by many authors [DK], [Ga], [Gr], [HSW], [N$_1$], [SN], [T].

Set $k_m = \boldsymbol{Q}(\zeta_m)$, where ζ_m is a primitive m-th root of unity. Let G be the galois group $\mathrm{Gal}(k_m/\boldsymbol{Q})$ of k_m over $\boldsymbol{Q}$. If k_m^+ is the maximal real subfield of k_m, then the ring $\boldsymbol{Z}_{k_m^+}$ of integers has always a power basis [Li], [W].

In this article we treat certain imaginary abelian subfields K with $[k_m : K] = 2$.

In the next section we consider the case that the conductor $m = 4p^n (n \geq 1)$ with a prime p and will show that the ring $\boldsymbol{Z}_K$ of any subfield K in k_m such that $[k_m : K] = 2$ has a power basis and it is generated by the Gauß period $\eta_H = \sum_{\rho \in H} \zeta_m^\rho$, where H is the subgroup of G corresponding to the field K. On the other hand, in the third section we prove that in the case that $m = 3p^n (n \geq 1)$ with a prime $p > 3$ and the subfield

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K which is distinct from $k_{m/3}$ and k_m^+, the ring $\boldsymbol{Z}_K$ of integers in K does not have a power basis.

Finally we will give another characterization of fields whose rings of integers do not have any power basis using the decomposition theory of ideals [N$_2$].

§2. Monogenic case

We start with the following theorems in which the rings of integers have power bases.

THEOREM 1. *Suppose $m = 2^n \geq 8$ and let K be the imaginary subfield of k_m distinct from $k_{m/2}$ such that $[k_m : K] = 2$. Then the ring $\boldsymbol{Z}_K$ of integers in K coincides with $\boldsymbol{Z}[\eta]$, where η is the Gauß period $\zeta_m - \zeta_m^{-1}$ and the absolute value of the field discriminant of K is equal to $2^{(n-1)\phi(2^{n-1})-1}$.*

Proof. Let $G = \mathrm{Gal}(k_m/\boldsymbol{Q}) = \langle\tau\rangle \times \langle\sigma\rangle$ with $\tau^2 = e = \sigma^s$, $s = \phi(m)/2 = 2^{n-2}$ and $\zeta_m^\tau = \bar{\zeta}_m$, $\zeta_m^\sigma = \zeta_m^5$, where $\bar{\alpha}$ means the complex conjugate of a number α and $\phi(\cdot)$ denotes the Euler function. Then $k_{m/2}$, $\boldsymbol{Q}(\zeta_m + \zeta_m^{-1})$ and K are subfields fixed by the subgroups $\langle\sigma^{s/2}\rangle$, $\langle\tau\rangle$ and $H = \langle\sigma^{s/2}\tau\rangle$ respectively. Then K is generated by the Gauss period $\eta = \sum_{\rho\in H}\zeta_m^\rho = \zeta_m - \zeta_m^{-1}$.

We see that $\boldsymbol{Z}_{k_m} = \boldsymbol{Z}[\zeta_m] = \boldsymbol{Z}_K[\zeta_m]$. Then, since $5^{2^{n-1}} \not\equiv -1 \pmod 4$, the relative different $\mathfrak{d}_{k_m/K}$ is given by

$$\left(\zeta_m - \zeta_m^{\sigma^{s/2}\tau}\right)\boldsymbol{Z}_{k_m} = (1-\zeta_m^2)\boldsymbol{Z}_{k_m} = \mathfrak{L}^2,$$

where $\mathfrak{L}$ is the ramified prime ideal $(1-\zeta_m)$ of k_m over 2. From this, it follows that

$$|d(K)| = \sqrt{|d(k_m)|/2^2} = 2^{s(n-1)-1}.$$

On the other hand, by $G/H = \left\{\sigma^j H; 0 \leq j < s\right\}$, the different $\mathfrak{d}_K(\eta)$ of η is given by

$$\prod_{j=1}^{s-1}(\eta - \eta^{\sigma^j}) = \prod_{j=1}^{s-1}\left\{\zeta_m\left(1-\zeta_m^{\sigma^j-1}\right)\left(1+\zeta_m^{-\sigma^j-1}\right)\right\}.$$

Since we observe that

$$\left\{\zeta_m^{\sigma^j},\ -\zeta_m^{-\sigma^j};\ 0 \leq j < s\right\} = \left\{\zeta_m^j;\ 0 < j < m,\ (j,m) = 1\right\},$$

$$\left\{\zeta_m^{\sigma^j-1},\ -\zeta_m^{-\sigma^j-1};\ 0 \leq j < s\right\} = \left\{\zeta_m^j;\ 0 \leq j < m,\ (j,m) \neq 1\right\},$$

we can put

$$X^m - 1 = \Phi_m(X)(X-1)\left(X + \zeta_m^{-2}\right) f(X),$$

where $\Phi_m(X)$ denotes the m-th cyclotomic polynomial and

$$f(X) = \prod_{j=1}^{s-1} \left\{ \left(X - \zeta_m^{\sigma^j - 1}\right)\left(X + \zeta_m^{-\sigma^j - 1}\right) \right\},$$

hence $m = \Phi_m(1)\left(1 - \zeta_m^{2s-2}\right) f(1)$. Then we obtain

$$\mathfrak{d}_K(\eta) \cong f(1) \cong 2^{n-1}/\mathfrak{L}^2,$$

namely

$$|d_K(\eta)| = 2^{s(n-1)-1}.$$

Here the symbol $\alpha \cong \beta$ or $\alpha \cong \mathfrak{A}$ onwards means $(\alpha) = (\beta)$ or $(\alpha) = \mathfrak{A}$ as ideals for numbers α, β and an ideal $\mathfrak{A}$, respectively.

THEOREM 2. *Suppose that $m = 4p^n$, where p is an odd prime and let K be the imaginary subfield of k_m distinct from $k_{m/4}$ with $[k_m : K] = 2$. Then the ring $\boldsymbol{Z}_K$ of integers in K coincides with $\boldsymbol{Z}[\eta]$, where η is the Gauß period $\zeta_m - \zeta_m^{-1}$ and the absolute value of the field discriminant of K is equal to $2^{\phi(p^n)} p^{n\phi(p^n) - p^{n-1} - 1}$.*

Proof. Let $G = \langle\tau\rangle \times \langle\sigma\rangle$ with $\zeta_4^\tau = \bar{\zeta}_4$, $\zeta_{m/4}^\tau = \zeta_{m/4}$ and $\zeta_4^\sigma = \zeta_4$, $\zeta_{m/4}^\sigma = \zeta_{m/4}^r$, where r is a primitive root modulo p^n. We have three subfields $k_{m/4}, k_m^+$ and K of degree $\phi(p^n)$ whose galois groups are $\langle\tau\rangle, \langle\sigma^s\tau\rangle$ and $H = \langle\sigma^s\rangle$ with $s = \phi(m/4)/2$ respectively. Denote ζ_4 by ι and $\zeta_{m/4}$ by ζ. For $\zeta_m = \iota\zeta$, let $\eta = \sum_{\rho\in H} \zeta_m^\rho = \iota\zeta + \iota\zeta^{-1} = \zeta_m - \zeta_m^{-1}$ be the Gauß period.

As in the proof of Theorem 1, since $\boldsymbol{Z}_{k_m} = \boldsymbol{Z}_K[\zeta_m]$, the relative different $\mathfrak{d}_{k_m/K}$ is given by

$$\left(\zeta_m - \zeta_m^{\sigma^s}\right) \boldsymbol{Z}_{k_m} = \iota(\zeta - \zeta^{-1}) \boldsymbol{Z}_{k_m} = \mathfrak{P},$$

where $\mathfrak{P}$ is the ramified prime ideal $(1-\zeta)$ of $k_{m/4}$ over p. Then

$$|d(K)| = \sqrt{|d(k_m)| / \mathrm{N}_{k_m}(\mathfrak{d}_{k_m/K})} = 2^{2s} p^{2ns - (m/4p) - 1}.$$

On the other hand, by $G/H = \left\{\sigma^j H,\ \sigma^j\tau H; 0 \le j < s\right\}$, the different $\mathfrak{d}_K(\eta)$ of η is given by

$$(\eta-\eta^\tau)\prod_{j=1}^{s-1}\left\{(\eta-\eta^{\sigma^j})(\eta-\eta^{\sigma^j\tau})\right\}$$
$$= (\iota/\zeta)^{2(s-1)}2\iota(\zeta+\zeta^{-1})\prod_{j=1}^{s-1}\left\{(\zeta^2-\zeta^{2\sigma^j})(\zeta^2-\zeta^{-2\sigma^j})\right\}.$$

Since we observe that

$$\left\{\zeta^{2\sigma^j},\ \zeta^{-2\sigma^j};\ 0\le j<s\right\} = \left\{\zeta^j;\ 0<j<m/4,\ (j,m/4)=1\right\},$$

we can put

$$\Phi_{m/4}(X) = (X-\zeta^2)\left(X-\zeta^{-2}\right)f(X),$$

where

$$f(X) = \prod_{j=1}^{s-1}\left(X-\zeta^{2\sigma^j}\right)\left(X-\zeta^{-2\sigma^j}\right),$$

hence $f(\zeta^2) = \Phi'_{m/4}(\zeta^2)\left(\zeta^2-\zeta^{-2}\right)^{-1}$. Then we obtain

$$\mathfrak{d}_K(\eta) \cong 2\Phi'_{m/4}(\zeta^2)/\left(\zeta-\zeta^{-1}\right) \cong 2p^n\mathfrak{P}^{-p^{n-1}-1},$$

namely

$$|d_K(\eta)| = \mathrm{N}_K\mathfrak{d}_K(\eta) = 2^{2s}p^{2ns}\cdot p^{-p^{n-1}-1} = 2^{2s}p^{2ns-m/(4p)-1}.$$

Therefore we obtain $|d(K)| = |d_K(\eta)|$. This completes the proof of Theorem 2.

Remark 1. Using the same way as in [W. Proposition 2.16.], we can give a simple proof of monogenesis of imaginary subfields once we know that they are generated by the Gauß period $\zeta_m-\zeta_m^{-1}$. Our methods of proofs for Theorem 1 and Theorem 2 which give a criterion to $\boldsymbol{Z}_K = \boldsymbol{Z}[\zeta_m-\zeta_m^{-1}]$ can be applied to investigate non-monogenic phenomena in Theorem 3.

§3. Non-Monogenic case

We claim that the ring $\boldsymbol{Z}_{k_m^-}$ of integers in an imaginary field k_m^- with $[k_m : k_m^-] = 2$ is non-monogenic. Contrary to the theorems in the previous section, the Gauß period does not generate a power basis.

THEOREM 3. *Suppose $m = 3p^n$, where p is a prime > 3, and K be the imaginary subfield of k_m distinct from $k_{m/3}$ with $[k_m : K] = 2$. Then the ring $\boldsymbol{Z}_K$ of integers in K does not have a power basis.*

Proof. Let $\omega = \zeta_3$, $\zeta = \zeta_{m/3}$. Then $\zeta_m = \omega \cdot \zeta$. For a cyclotomic field $k_m = \boldsymbol{Q}(\zeta_m)$, let

$$G = \mathrm{Gal}(k_m/\boldsymbol{Q}) = \langle \tau \rangle \times \langle \sigma \rangle$$

be the galois group with $\tau^2 = e = \sigma^{\phi(m/3)}$ and $\omega^\tau = \bar{\omega}$, $\omega^\sigma = \omega$, $\zeta^\tau = \zeta$, $\zeta^\sigma = \zeta^r$, where r is a primitive root modulo $p^n = m/3$. Then $\zeta_m^\tau = \bar{\omega} \cdot \zeta$, $\zeta_m^\sigma = \omega \cdot \zeta^r$.

For $s = \phi(m/3)/2$, let $H = \langle \sigma^s \rangle$ be the subgroup of G corresponding to K and $\eta = \sum_{\rho \in H} \zeta^\rho = \omega(\zeta + \zeta^{-1})$ be the Gauß period. Then $K = \boldsymbol{Q}(\eta)$. Since $\boldsymbol{Z}_K = \boldsymbol{Z}_{k_3} \boldsymbol{Z}^+_{k_{m/3}} = \omega \boldsymbol{Z}[\gamma] + \omega^\tau \boldsymbol{Z}[\gamma]$, any $\xi \in \boldsymbol{Z}_K$ can be written as $\xi = \omega R + \omega^\tau S$ with $R, S \in \boldsymbol{Z}[\gamma]$, where $\gamma = \zeta + \zeta^{-1}$. Then by $G/H = \left\{ \sigma^j H,\ \sigma^j \tau H; 0 \le j < s \right\}$, the different $\mathfrak{d}_K(\xi)$ of ξ is given by

$$\begin{aligned} &(\xi - \xi^\tau) \prod_{j=1}^{s-1} \left\{ (\xi - \xi^{\sigma^j})(\xi - \xi^{\sigma^j \tau}) \right\} \\ &\quad = (\omega - \omega^\tau)(R - S) \prod_{j=1}^{s-1} \left\{ (\xi - \xi^{\sigma^j \tau}) \right\} \prod_{j=1}^{s-1} \left\{ \omega(R - R^{\sigma^j}) + \omega^\tau (S - S^{\sigma^j}) \right\}. \end{aligned}$$

Here, we observe that $T - T^\rho$ is always divisible by $\gamma - \gamma^\rho = \zeta - \zeta^\rho + \zeta^{-1} - \zeta^{-\rho}$, which is further divisible by $\mathfrak{P}$, if $T \in \boldsymbol{Z}[\gamma]$ and $\rho \in G$, where $\mathfrak{P}$ is the ramified prime ideal $(1 - \zeta)$ of $k_{m/3}$ over p. Therefore $\mathfrak{d}_K(\xi)$ is a multiple of

$$(1 - \omega)(\xi - \xi^{\sigma\tau}) \prod_{j=1}^{s-1} \left(\gamma - \gamma^{\sigma^j} \right) = (1 - \omega) \left(\xi - \xi^{\sigma\tau} \right) \mathfrak{d}_{k^+_{m/3}},$$

namely $d_K(\xi)$ is a multiple of

$$N_K \left(\xi - \xi^{\sigma\tau} \right) 3^s d \left(k^+_{m/3} \right) = N_K \left(\xi - \xi^{\sigma\tau} \right) d(K).$$

Moreover, by the observation above, we have:

(i) If $R = S^\sigma$, then $\xi - \xi^{\sigma\tau} = \omega^\tau \left(S - S^{\sigma^2}\right) \in \mathfrak{P}$;

(ii) If $S = R^\sigma$, then $\xi - \xi^{\sigma\tau} = \omega \left(R - R^{\sigma^2}\right) \in \mathfrak{P}$;

(iii) If $R - S^\sigma = S - R^\sigma$, then $2\left(\xi - \xi^{\sigma\tau}\right) = -(R+S) + (R+S)^\sigma \in \mathfrak{P}$;

(iv) If $R - S^\sigma = R^\sigma - S$, then $\left(\xi - \xi^{\sigma\tau}\right) = (\omega - \omega^\tau)(R - S^\sigma) \in (1-\omega)$;

(v) Otherwise, as R, S are totally real, we have

$$\begin{aligned}|N_K\left(\xi - \xi^{\tau\sigma}\right)| &= \left|N_{k^+_{m/3}}\left(\left(R - S^\sigma\right)^2 - \left(R - S^\sigma\right)\left(S - R^\sigma\right) + \left(S - R^\sigma\right)^2\right)\right| \\ &> \left|N_{k^+_{m/3}}\left(\left(R - S^\sigma\right)\left(S - R^\sigma\right)\right)\right| \\ &\geq 1.\end{aligned}$$

This implies that $|N_K\left(\xi - \xi^{\tau\sigma}\right)| > 1$ whenever $\xi - \xi^{\tau\sigma} \neq 0$. Hence, we find that $|d_K(\xi)| > |d(K)|$ if $d_K(\xi) \neq 0$.

Remark 2. As in the previous section, since $\boldsymbol{Z}_{k_m} = \boldsymbol{Z}_K[\zeta_m]$, the relative different $\mathfrak{d}_{k_m/K}$ is given by

$$\left(\zeta_m - \zeta_m^{\sigma^s}\right)\boldsymbol{Z}_{k_m} = \mathfrak{P}\boldsymbol{Z}_{k_m}.$$

Then

$$|d(K)| = \sqrt{|d(k_m)|/\mathrm{N}_{k_m}(\mathfrak{d}_{k_m/K})} = 3^s p^{2ns-(m/3p)-1}.$$

The following is slightly generalized from [N$_2$] owing to a remark from L. Washington.

Proposition. *Let K be a galois extension of degree $n > 2$ over $\boldsymbol{Q}$ and ℓ be a prime number of ramification index e and relative degree f for $K/\boldsymbol{Q}$. If either $e\ell^f < n$ or $f > 1$, $e\ell^f \leq n + e - 1$, then $\boldsymbol{Z}_K$ does not have a power basis.*

Proof. Let α be a primitive element of K in $\boldsymbol{Z}_K$. Let the prime ideal decomposition of ℓ in the field K be

$$\ell \cong \prod \mathfrak{L}^e.$$

For any prime ideal $\mathfrak{L}$, we have

$$\alpha^{\mathrm{N}_K \mathfrak{L}} \equiv \alpha \bmod \mathfrak{L}.$$

Then by

$$\alpha^{\mathrm{N}_K \mathfrak{L}} \equiv \alpha \pmod{\prod \mathfrak{L}},$$

we see that

$$(\alpha^{\mathrm{N}_K \mathfrak{L}} - \alpha)^e \equiv 0 \pmod{\ell}.$$

Thus if $e\mathrm{N}_K \mathfrak{L} = e\ell^f < n$, then certainly the number

$$\beta = \ell^{-1}(\alpha^{\mathrm{N}_K \mathfrak{L}} - \alpha)^e = (1/\ell)\alpha^{e\ell^f} \pm \cdots \pm (1/\ell)\alpha^e$$

is in $\boldsymbol{Z}_K$ but outside of $\boldsymbol{Z}[\alpha]$. If $(\alpha, \ell) = 1$, $e\ell^f \leq n + e - 1$, then $\alpha^{-e}\beta \in \boldsymbol{Z}_K$ but $\notin \boldsymbol{Z}[\alpha]$. If $(\alpha, \ell) \neq 1$ and $\boldsymbol{Z}_K = \boldsymbol{Z}[\alpha]$, then $\alpha \equiv 0 \,(\mathrm{mod}\, \mathfrak{L})$ for a certain $\mathfrak{L}$, hence for any integer $\xi = b_0 + b_1\alpha + \cdots + b_{n-1}\alpha^{n-1} \in \boldsymbol{Z}_K$, we have $\xi \equiv b_0 \pmod{\mathfrak{L}}$, namely $f = 1$, which contradicts the hypothesis. Thus there exists an integer of K, but outside of $\boldsymbol{Z}[\alpha]$.

Example. Consider for the case of conductor $m = |5 \cdot (-3)| = 15$ a subfield $K = \boldsymbol{Q}(\sqrt{5}, \sqrt{-3})$ of $k_{15} = \boldsymbol{Q}(\zeta_{15})$ with $[k_{15} : K] = 2$. Since the prime number 2 splits in $\boldsymbol{Q}(\sqrt{-15})$ and $\mathfrak{L}$ is inert in $K/\boldsymbol{Q}(\sqrt{-15})$ for a prime ideal $\mathfrak{L}|2$, the ring $\boldsymbol{Z}_K$ of integers has no power basis by Proposition. Using the Gauß period $\eta = \zeta_3(\zeta_5 + \zeta_5^{-1})$, we have $K = \boldsymbol{Q}(\eta)$. Then the non-monogenesis of the ring $\boldsymbol{Z}_K$ is confirmed by Theorem 3, too. The other examples of prototype are shown in [SN].

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