p-Divisible Groups and the Chromatic Filtration

January 20, 2010

1 Chromatic Homotopy Theory

Some problems in homotopy theory involve studying the interaction between generalized cohomology theories. This is usually performed by examining the properties of certain representing objects called spectra, their homotopy groups, and morphisms between them. In particular, the setup for most computation in modern algebraic topology, including work with the Adams and the Adams-Novikov spectral sequences, is carried out in this language. For example, integral K-theory and singular cohomology are given by the spectrum $\mathbb{Z} \times BU$ and the Eilenberg-MacLane spectrum $H\mathbb{Z}$, respectively. Another important spectrum is MU, the spectrum corresponding to complex cobordism.

As in other areas, integral computations are difficult, and more understanding is looked for by examining the local or complete cases. At the prime p there is an arrangement of spectra, called the chromatic filtration, that provides a sequence of approximations of p-adic stable cohomotopy, the first two stages of which are p-adic rational cohomology and p-adic K-theory. This filtration reflects the deep connection between underlying structure in algebraic topology and the moduli space of formal group laws. This connection was first formulated by Novikov and Quillen, who noticed that the complex cobordism of a point was the Lazard ring $L = MU^*(pt)$, which classifies formal group laws. A classifying map $MU^*(pt) \to R$ to some commutative ring R that satisfies an additional condition called Landweber exactness can be used to generate a new cohomology theory from MU with coefficient ring R, at least over finite complexes X; for example, K-theory can be constructed in this way from the multiplicative formal group law: $K^*(X) = K^*(pt) \otimes_{MU^*(pt)} MU^*(X)$.

Over a separably closed field of characteristic p, formal group laws are classified up to isomorphism by their *height*, with height one corresponding to the multiplicative formal group law, and height infinity corresponding to the additive formal group law. Lubin-Tate deformation theory studies lifts of a formal group law H_n of height n over a perfect field k of characteristic p to mixed-characteristic complete local rings with residue field k. In particular, there exists a complete local ring E_n, called the Lubin-Tate space, with a formal group law F_n defined over it, which is a lift of H_n and is universal among all such lifts. The classifying map for this formal group law is Landweber exact, and can be used to construct a generalized cohomology theory known as the n-th Morava E-theory, $E_n^*(X) := E_n^* \otimes_{MU^*(pt)} MU^*(X)$ (with appropriate grading). A related cohomology theory, known as Morava K-theory, $K(n)$, with the localized coefficient ring $\mathbb{F}_p[v_n, v_n^{-1}]$, is also obtained in this way. The E_n's contain information about the *layers* in the chromatic tower, which are usually taken to refer to the $K(n)$-localization of the sphere spectrum. The first two Morava E-theories are familiar: E_1 is p-adic K-theory and E_2 is a form of elliptic cohomology.

The hope of locally performing computations level-wise, examining transchromatic phenomena, and then iteratively climbing the chromatic tower has helped make the Morava E-theories a central focus of study in algebraic topology. One of the most well-known attempts to understand these higher cohomology theories was an endeavor begun by Hopkins, Kuhn, and Ravenel in the 1990s to generalize to higher heights Atiyah's character theorem, which identifies the K-theory of the classifying space of a group G with the representation ring of G, completed at the augmentation ideal of virtual representations. (HKR01) uses concepts of Tate modules and Drinfel'd level structures to give a theory of higher characters for Morava E_n's. A more recent sequence of articles by Torii constructs a generalized Chern character by defining a transchromatic map from E_{n+1} to an extension of E_n and proving that this map is rationally an isomorphism. In addition, Torii shows that the generalized Chern character is compatible with the HKR generalized character theory, appropriately extended.

There are two points of underlying interest in these results. The first is that both works gain deeper understanding of cohomology theories by looking not just at the corresponding formal groups, but at the structure of the associated p-divisible groups, their hom-sets, and their behavior upon base change. Formal groups correspond exclusively to *connected* p-divisible groups of a fixed height. In general, p-divisible groups contain information about formal groups of different heights and can be disconnected with etale quotients. Thus, passing to the category of p-divisible groups can be thought of a true extension of the playing field. p-Divisible groups have been well-studied in their own right in algebraic geomety by Tate, Serre, Manin, among others, in connection with lifting of homomorphisms of abelian varieties. It should be possible to further harness the developed framework in examining chromatic phenomena.

The second point of interest is that both results are *equivariant* in nature: they are compatible with the action of the Morava stabilizer group S_n of (strict) automorphisms of H_n. Automorphisms of objects are important, and the the action of the Morava stabilizer group on the Lubin-Tate space E_n is very interesting in particular. For example, knowledge of the action of S_n on E_n gives a a spectral sequence computing the homotopy of the $K(n)$-localization of a spectrum X from the cohomology of the extended stabilizer group with coefficients in $E_n^*(X)$. Other applications include examining homotopy fixed point spectra of E_n under the action of all of S_n or just certain finite subgroups, which give the $K(n)$-local sphere and higher real K-theory, respectively.

However, right now computations in these realms seem just out of reach because, despite many efforts, little is known about the action of the stabilizer group explicitly. Getting more information about this action is of direct interest and may have immediate tangible resonance in stable homotopy theory. Now, as will be seen from the height-one case described below, the stabilizer group action on E_1 appears as a certain Galois action on a trivialized p-divisible group. Perhaps some similar statements can be formulated to hold in general. This would be very exciting, as it would give a more algebraic, galois-theoretic interpretation of the inherently topological action of the stabilizer group.

Combining these two observations gives the philosophical basis of the proposed project: we wish to examine the further information that p-divisible groups, together with an action of the Morava stabilizer group, can give about the chromatic filtration of stable homotopy theory. A more tangible and reassuring form of a project statement is given below after presenting the known case - the computations at height one.

2 The Picture at Height One. A Problem Statement

Examine height n=1, and work over $k = \mathbb{F}_p$. Morava E_1^* is p-adic K-theory, and the Lubin-Tate space is the ring of Witt vectors over k, $E_1 = W(k) = \mathbb{Z}_p$. The associated formal group law G_1 is deduced from the tensor product of vector bundles to be the multiplicative formal group law with p-series $[p^r](t) = 1 - t^{p^r}$. Note that in this case the formal power series for $[p^r](t)$ coincides with its Weierstrass polynomial.

HKR character theory gives a map from the E_n cohomology of classifying spaces of finite groups to class functions on commuting n-tuples of p-power elements. The HKR character map becomes an injection over the Drinfel'd ring D_1, and an isomorphism over the HKR ring L_1 - a localization of D_1 at the set of Chern classes. At height one, D_1 is the direct limit of the cohomology of p-power cyclic groups of rank one. As in the general case, this rank is determined by the structure of the abelian group of p^r-torsion points, which is of rank equal to the height n. D_1 can be shown to be the extension of the p-adic integers by p-power roots of unity, and $L_1 = p^{-1}D_1$ is the corresponding colimit of cyclotomic field extensions of Q_p.

The HKR character map χ_{hkr} has been computed for n=1 explicitly. For a finite group G,

$$\chi_{hkr} : K(BG) \longrightarrow \prod_{g \in G_p} L_1$$
$$\xi \mapsto (\chi_{cl}(g)(\xi))_{g \in G_p}$$

Here G_p is again the set of p-power elements of G and χ_{cl} denotes the classical trace character from representation theory. Note that we are thinking of ξ both as a vector bundle and as the corresponding monodromy representation of G.

This map is equivariant with respect to the action of the Morava stabilizer group S_1, and the action of S_1 on both sides is known. At $n = 1$, S_1 is the group of units the p-adics, $S_1 = \mathbb{Z}_p^\times$. An element $\sigma \in S_1$ acts on the complex representation ring by sending a representation ξ to the representation $g \to \xi(g^\sigma)$. The same element acts on the right hand side through an action on the HKR ring L_1 by fixing $\mathbb{Z}_p$ and raising p-power roots of unity to the power $\sigma \in \mathbb{Z}_p^\times$. This statement is the p-adic form of a familiar theorem from representation theory.

Now, for the p-divisible group side of the story. The p-divisible group associated to E_1^* over $E_1 = \mathbb{Z}_p$ is the sequence of finite flat group schemes

$$\mathbb{G} = \{\mathbb{G}_r, i_r\} = \{\mathrm{Spec}(\mathbb{Z}_p[t]/(1 - t^{p^r})), i_r\},$$

where i_r is induced by inclusion of torsion points.

The additional structure carried by $\mathbb{G}$ is demonstrated by examining its behavior over an appropriate tower-like filtration of the base scheme Spec(E_1) = Spec($\mathbb{Z}_p$). The idea here is to filter the Lubin-Tate space by quotients by increasing invariant ideals, and to branch out each successive quotient by embedding it into a localization, followed by the quotient field, and then the separable closure. For n=1, this picture is simple:

$$\begin{array}{ccc} E_1 = \mathbb{Z}_p & \longrightarrow & p^{-1}(E_1) = \mathbb{Q}_p \\ \downarrow & & \\ E_1/(p) = \mathbb{F}_p & & \end{array}$$

Over $\mathbb{Z}_p$, the rings representing each finite layer $\mathbb{G}_r$ are local rings with maximal ideal $(p, 1 - t)$, in particular, they are connected rings, so each $\mathbb{G}_r$ is a connected group scheme, as it should be. However, over $\mathbb{Q}_p$, a field of characteristic 0, the p-divisible group $\mathbb{G}$ becomes etale. It is represented by a product of cyclotomic unramified extensions of $\mathbb{Q}_p$ by p-power primitive roots of unity:

$$(\mathbb{G}_{\mathbb{Q}_p})_r = \mathrm{Spec}(\prod_{0 \le i \le r} \mathbb{Q}_p(\zeta_{p^i}))$$

This is a finite group scheme of total rank p^r, which is etale over $\mathbb{Q}_p$, and which becomes constant over $\mathbb{Q}_p(\zeta_{p^r})$. Taking colimits, $\mathbb{G}$ becomes the constant p-divisible group of height 1, $\mathbb{Q}_p/\mathbb{Z}_p$, over the field extension of the p-adic numbers, obtained by adjoining all p-power roots of unity (i.e., the HKR ring, L_1). Note that if we did not invert p, the universal place where $\mathbb{G}$ becomes constant is actually the Drinfel'd ring, D_1, and $L_1 = p^{-1}D_1$.

Now, a result from the theory of finite group schemes states that mapping to geometric points gives an equivalence of categories between finite group schemes over a field K of characteristic zero and finite Galois modules with an action of the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$. Here $(\mathbb{G}_{\mathbb{Q}_p})_r(\overline{\mathbb{Q}_p})$ is μ_{p^r}, the group of all p^r-th roots of unity, with group action being multiplication in the field. The Galois group of the intermediate (abelian) extension of $\mathbb{Q}_p$ by p-power roots of unity, $\mathbb{Z}_p^\times$, acts on this set by raising a root of unity to the corresponding unit-p-adic power. Note that, by coincidence or otherwise, this gives exactly the action of the Morava stabilizer group S_1 on the HKR ring L_1.

The picture at n=1 is nice enough to warrant further exploration. The proposed project is to attempt explicit computation to determine what happens at higher heights, beginning with n=2. That is, the goal is to take the described filtration of E_n by invariant ideals, embed each quotient in, successively, localizations, quotient fields, and separable closures, and then examine the behavior of the p-divisible group $\mathbb{G}_n$ upon horizontal and vertical base change (see diagram below). Particular questions include: what does $\mathbb{G}_n$ look like over each complete local ring at the trunk of the tower? On each branch, when does it become etale? Constant? What is the Galois action on the etale p-divisible groups, and can it be tied to the action of the Morava stabilizer group?

These questions were inspired by conversations with, and, to be precise, were in some form originally due to Ando, Miller, and Morava. Basis in the literature is traced to papers of Hopkins, Kuhn, Ravenel; Tate; and Torii.

3 First Steps. A Tower.

This section gives a more explicit description of the proposed filtration of E_n.

Work at height n, at a prime p. Fix a base field $k = \mathbb{F}_{p^n}$. Then $E_n = W(k)[[u_1, ..., u_{n-1}]]$. This is a complete local ring with maximal ideal $I = (p, u_1, ..., u_{n-1})$. Consider the chain of invariant ideals $\{I_i = (p, u_1, ..., u_{i-1})\}_{0 \le i \le n}$, where $I_0 = 0$ and $I_1 = p$. Successively modding out by I_i and branching out each quotient gives the following filtration of E_n.

$$
\begin{array}{l}
E_n/I_0 = E_n = W(k)[[u_1, ..., u_{n-1}]] \to p^{-1}(E_1) \to \mathrm{Quot}(E_1)^{sep} \\
\qquad \downarrow \\
E_n/I_1 = E_n/(p) = k[[u_1, ..., u_{n-1}]] \to u_1^{-1}k[[u_1, ..., u_{n-1}]] \to k((u_1))[[u_2, ..., u_{n-1}]] \to \mathrm{Quot}(E_n/I_1)^{sep} \\
\qquad \downarrow \\
\qquad ... \\
\qquad \downarrow \\
E_n/I_{n-1} = E_n/(p, u_1, ..., u_{n-2}) = k[[u_{n-1}]] \to u_{n-1}^{-1}k[[u_{n-1}]] = k((u_{n-1})) \to k((u_{n-1}))^{sep} \\
\qquad \downarrow \\
E_n/I_n = E_n/(p, u_1, ..., u_{n-1}) = k
\end{array}
$$

Note that only fields that appear on the first line (E_n/I_0) are of characteristic 0, while all fields below that are of characteristic $p > 0$. Also, the local rings starting from the second line (E_n/I_1) are equicharacteristic local rings, while the local rings on the first line are of mixed characteristic.

The study of p-divisible groups over complete discrete valuation rings with quotient fields of characteristic 0 is the classical case, as developed by Tate and Serre. Working over complete discrete valuation rings with quotient fields of characteristic $p > 0$ involves looking at works of Gross, who has some results for the equicharacteristic case. Working over more general local rings might involve expanding the framework used.