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The torsion-free rank of $\mathrm{Ext}^1(\mathrm{A},\mathbb{Z})$ and Whitehead's problem

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Abstract

The aim of this article is to give an alternative proof of the characterization of the torsion-free rank of $\mathrm{Ext}^1(A,\mathbb{Z})$ for countable torsion-free abelian groups A with torsion-free rank equals to 1, without using Stein's Theorem. This leads to the characterization of the torsion-free rank of $\mathrm{Ext}^1(A,\mathbb{Z})$ given in Eklof and Mekler's book *Almost Free Modules*. As a result, Whitehead's conjecture is verified for the case of countable groups.

Keywords: Whitehead's problem, countable abelian groups, Ext groups, divisible groups.

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1 Introduction

Whitehead's problem states that a commutative group A is free if and only if its associated group $\mathrm{Ext}^1(A,\mathbb{Z})$ is trivial. Groups satisfying the latter property are called Whitehead's groups, from here onwards *W-groups*.

Whitehead's problem appeared at the beginning of the fifties of the past century. In 1951 it was proved by Stein [7] that every countable W-group is free. For what concerns larger W-groups the first major result is due to Saharon Shelah [6]: in an unexpected turn of events in 1974, he proved that, at least for cardinality $\aleph_1$, Whitehead's Problem is undecidable on the basis of ZFC set theory, by showing that both the affirmative or negative answers to Whitehead's Problem are consistent with ZFC (Zermelo-Fraenkel set theory with Choice).

The article sticks to the case of countable abelian groups: we will follow Chase's ([1], 1963) approach, which uses homological algebraic tools and passes through the characterization of $\mathrm{Ext}^1(A,\mathbb{Z})$ in terms of a set of cardinals, called the *torsion-free rank* and *p-ranks*. The core of the article is Section 5: we offer an alternative proof of the characterization of the torsion-free rank of $\mathrm{Ext}^1(A,\mathbb{Z})$ for a countable group A of torsion-free rank equal to 1. The overall architecture of the proof is similar to the one presented for instance in the book *Almost Free Modules* ([4, XII, Thm. 4.1]). However, the methods stay away from set theory and our proof does not use Stein's Theorem (the countable case of Whitehead's Problem), but rather implies it (Corollary 5.11). For completeness of the discussion we include the proof of Chase's Characterization, which states that the torsion free rank of $\mathrm{Ext}^1(A,\mathbb{Z})$ is equal to $2^{\aleph_0}$, whenever A is a countable torsion-free group which is not free.

In section 2, we introduce notation and recall the tools of abstract and homological algebra used. The definition of the ranks of a group is contained in the third section, and so is Pontryagin's Criterion. The reduction to the case when A is torsion-free (under which assumption the group $\mathrm{Ext}^1(A,\mathbb{Z})$ is injective, see section 6) is contained in the fourth section.

The main references are Fuchs's book *Infinite Abelian Groups* and Eklof and Mekler's book *Almost Free Modules, Set-theoretic Methods*, whose content the article stays close to.

2 Notation and Recollections

Unless otherwise specified, in what follows "ring" will be a shorthand for "commutative ring with identity" and "group" will be a shorthand for "abelian group". The group of homomorphisms between two groups A and B will be denoted by $\mathrm{Hom}(A,B)$: instead, if we consider morphisms of R modules with R a ring different from $\mathbb{Z}$ we will use the notation Hom_R. Given a group A, for each prime number p, we denote by A_p the subgroup of A

$$\{a \in A : \exists k \in \mathbb{N} \ s.t. \ ord(a) = p^k\}$$

where $ord(a)$ (the *order* of a) is the least $n \in \mathbb{N} \setminus \{0\}$ such that $n \cdot a = 0$. The subgroup A_p is called the *p-component* of A. In what follows, we will make largely use of p-components of the quotient group $\mathbb{Q}/\mathbb{Z}$ (using addition of rational numbers as group operation), which go under the name of *Prüfer p-groups* and are denoted by $\mathbb{Z}(p^\infty)$. In particular in Section 5, we will use the connection between the Prüfer p-group and $\mathbb{Z}_p$, the group of p-adic integers, stated in the following theorem.

Theorem 2.1. *[5, Prop. 5.26 - Ex. 5.20] The group of endomorphisms of $\mathbb{Z}(p^\infty)$ is isomorphic to the group of p-adic integers $(\mathbb{Z}_p, +)$, i.e. $End_{\mathbb{Z}}(\mathbb{Z}(p^\infty)) \cong \mathbb{Z}_p$.*

Another statement which we will make use of, and which involves Prüfer groups, is the following.

Lemma 2.2. *[2, Cor. 43.4] Let $T(A)$ be the torsion subgroup of a group A. Then*

$$Hom(T(A), \mathbb{Q}/\mathbb{Z}) \cong \prod_p Hom(T_p(A), \mathbb{Z}(p^\infty))$$

where $T_p(A)$ is the p-components of $T(A)$ as p ranges over the primes.

A recurrent notion in this article is that of a divisible group: more generally, an R-module M is said to be *divisible* if

$$M = r \cdot M = \{r \cdot m : m \in M\} \quad \forall r \in R \setminus \{0\}.$$

Example 2.3. Both $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are divisible groups. Also Prüfer groups $\mathbb{Z}(p^\infty)$ are examples of divisible groups.

Remark 2.4. Over a commutative PID R, divisible modules are the same as *injective* modules [5, Corollary 3.35], i.e. modules I for which $Hom_R(-, I)$ is an exact functor. Hence, since abelian groups are $\mathbb{Z}$-modules, an equivalent condition for a group A to be divisible is being injective.

Example 2.5. The condition of R being a PID is necessary, otherwise there could be divisible R-modules which are not injective. For example, consider $R = \mathbb{Z}[x]$ and its field of fractions $\mathbb{Q}(x)$: then the quotient $\mathbb{Q}(x)/\mathbb{Z}[x]$ is a divisible $\mathbb{Z}[x]$-module but it is not injective.

Let A and B be abelian groups: viewing them as $\mathbb{Z}$-modules, we consider the right derived functors of $\mathrm{Hom}(A; -)$, denoted by $\mathrm{Ext}^i(A; -)$ for $i \geq 0$, and the left derived functors of $\mathrm{Hom}(-; B)$, denoted by $\mathrm{Ext}^i(-; B)$. By virtue of the so-called *Balance of Ext* Theorem (whose proof can be found in Lang's book [3, Cor. XX.8.5] for instance), there is an isomorphism

$$\mathrm{Ext}^i(A; -)(B) \cong \mathrm{Ext}^i(-; B)(A), \quad \forall i \geq 0,$$

which leads to use the same notation $\mathrm{Ext}^i(A; B)$ for both. Observe that this double characterization allows to use injective resolutions for B as well as projective ones for A in order to compute $\mathrm{Ext}^i(A; B)$.

Remark 2.6. Let A, B be abelian groups. Since $\mathbb{Z}$ is a commutative PID, the groups $\mathrm{Ext}^i(A, B)$ are all trivial for $i \geq 2$. Therefore, given a short exact sequence of abelian groups

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0,$$

the long exact sequence induced by $\mathrm{Hom}(-; B)$ is then

$$0 \longrightarrow \mathrm{Hom}(A'', B) \longrightarrow \mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(A', B) \longrightarrow \mathrm{Ext}^1(A'', B) \longrightarrow \mathrm{Ext}^1(A, B) \longrightarrow \mathrm{Ext}^1(A', B) \longrightarrow 0.$$

The same holds for the functor $\mathrm{Hom}(A, -)$. For a detailed discussion of these basic but very deep results of Homological Algebra see Rootman's book *An Introduction of Homological Algebra* (in particular Chapter 6).

Lastly, the following property $\mathrm{Ext}^1(-, B)$ follows from the one of $\mathrm{Hom}(-, B)$.

Theorem 2.7. *[2, Thm. 52.2] Let $\{A_i\}_{i \in I}, B$ be all abelian groups. Then*

$$Ext^1\left(\bigoplus_{i \in I} A_i, B\right) \cong \prod_{i \in I} Ext^1(A_i, B).$$

3 The ranks of a group

In this section we focus on subgroups of abelian groups that are direct sums of cyclic groups. We will use subgroups which are maximal with respect to this property to define cardinal numbers depending only on A. This will lead to the definition of *ranks* of A, which extends to groups the notion of dimension for vector spaces.

We next define the ranks and review important theorems that will allow us to understand the structure of $\mathrm{Ext}^1(A, \mathbb{Z})$. The omitted proofs can be found in Chapters 16, 19 and 23 of [2].

We recall that by an *independent* system $L = \{a_i\}_{i \in I}$ of non-zero elements of a group A one means that for each finite subsystem of $\{a_1, \dots a_n\} \subseteq L$

$$\sum_{i=1}^{k} n_i a_i = 0 \ (n_i \in \mathbb{Z}) \implies n_i a_i = 0 \ \ \forall i \in \{1, \dots, k\}.$$

Given an independent system L, an element $g \in A \setminus \{0\}$ is *dependent* on L if there exist $m, n_1, \dots, n_k \in \mathbb{Z}$ and $a_1, \dots, a_k \in L$ such that

$$mg = \sum_{i=1}^{k} n_i a_i \neq 0.$$

An independent system M of A is *maximal* if there is no independent system in A containing M properly.

Definition 3.1. Given a group A, let M_0 be an independent system of A containing only elements of infinite order, maximal with respect to this property[1]. The *torsion-free rank* of A, denoted by $r_0(A)$, is the cardinality of M_0.

Analogously, for p ranging over the prime numbers, define the *p-rank* $r_p(A)$ of A as the cardinality of an independent system M_p which contains only p-elements, namely the elements whose order is a power of p.

[1] Zorn's Lemma ensures its existence.

It happens that for any group A the cardinals $r_0(A)$ and $r_p(A)$ are independent respectively of the maximal systems M_0 and M_p chosen to compute them.

Theorem 3.2. *[2, Thm. 16.3] For any group A the cardinals $r_0(A)$ and $r_p(A)$ do not depend on the chosen maximal independent systems M_0 and M_p, for any p: hence we have a well defined notions of torsion-free rank and p-ranks for A.*

Example 3.3. In order to get a little more acquainted with the definitions of ranks, let us compute them for $\mathbb{Q}$ and for Prüfer groups $\mathbb{Z}(p^\infty)$.

For the field $\mathbb{Q}$, since any element $a/b \in \mathbb{Q}^*$ is dependent on $\{1\}$ ($b\cdot a/b = a\cdot 1$), a maximal independent system M_0 is $\{1\}$. Thus the torsion-free rank $r_0(\mathbb{Q})$ is equal to 1: moreover, since there is no-torsion in $\mathbb{Q}$, then all the p-ranks $r_p(\mathbb{Q})$ are null.

Instead, since the group $\mathbb{Z}(p^\infty)$ is defined as the p-component of a group, its torsion-free rank $r_0(\mathbb{Z}(p^\infty))$ and q-ranks $r_q(\mathbb{Z}(p^\infty))$ with $q \neq p$ are trivial. However the p-rank $r_p(\mathbb{Z}(p^\infty))$ is equal to 1: indeed $\{1/p\}$ is a maximal system of p-elements, since any non-null element $a/p^k \in \mathbb{Z}(p^\infty) = (\mathbb{Q}/\mathbb{Z})_p$ with $(a,p) = 1$ is dependent on $1/p$ ($p^{k-1}\cdot a/p^k = a\cdot 1/p \neq 0$).

In terms of ranks a useful statement can be formulated, which establishes a criterion for determining whether a countable group is free.

Theorem 3.4. *[2, Thm. 19.1] (Pontryagin's Criterion) A countable torsion-free group A is free if and only if every finite rank subgroup is free.*

The importance of the ranks for a group is linked to the existence of a complete classification of divisible groups in terms of the torsion-free rank and p-rank, provided by the following theorem.

Theorem 3.5. *[2, Thm. 23.1] Any divisible group A is a direct sum of copies of Prüfer groups $\mathbb{Z}(p^\infty)$ and copies of $\mathbb{Q}$. More precisely, one has the decomposition*

$$A \cong \mathbb{Q}^{\oplus r_0(A)} \oplus \bigoplus_p \mathbb{Z}(p^\infty)^{\oplus r_p(A)}$$

where p runs over the primes.

4 The Whitehead problem: reduction to the torsion-free case

Recall that a W-group is a group A such that $\mathrm{Ext}^1(A,\mathbb{Z}) = 0$. If A is a free abelian group, then it is projective and so a projective resolution for A is

$$0 \longrightarrow A \xrightarrow{id} A \longrightarrow 0.$$

This means that $\mathrm{Ext}^1(A,\mathbb{Z})$ is trivial. Therefore free abelian groups are W-groups. The Whitehead's problem asks whether it is possible to find a W-group which is not free. In any case, if a group A has $\mathrm{Ext}^1(A,\mathbb{Z})$ trivial, then it cannot have torsion.

Proposition 4.1. *Any W-group is torsion-free.*

Proof. Let A be a group and $T(A)$ its torsion subgroup: pick and element $a \in T(A)$ and consider the cyclic subgroup $\langle a\rangle_A$ generated by a. Applying the functor $\mathrm{Hom}(-,\mathbb{Z})$ to the short exact sequence

$$0 \longrightarrow \langle a\rangle \longrightarrow A \longrightarrow A/\langle a\rangle \longrightarrow 0,$$

by Remark 2.6 we obtain the following long exact sequence in cohomology

$$0 \longrightarrow \mathrm{Hom}(A/\langle a\rangle,\mathbb{Z}) \longrightarrow \mathrm{Hom}(A,\mathbb{Z}) \longrightarrow \mathrm{Hom}(\langle a\rangle,\mathbb{Z}) \longrightarrow \mathrm{Ext}^1(A/\langle a\rangle,\mathbb{Z}) \longrightarrow \mathrm{Ext}^1(A,\mathbb{Z}) \longrightarrow \mathrm{Ext}^1(\langle a\rangle,\mathbb{Z}) \longrightarrow 0.$$

In particular, this gives a surjective map $\mathrm{Ext}^1(A,\mathbb{Z}) \twoheadrightarrow \mathrm{Ext}^1(\langle a\rangle,\mathbb{Z})$: therefore if A is a W-group, $\mathrm{Ext}^1(A,\mathbb{Z}) = 0$ and so is $\mathrm{Ext}^1(\langle a\rangle,\mathbb{Z})$. Since the subgroup $\langle a\rangle$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ (where n is the order of a), we have $\mathrm{Ext}^1(\langle a\rangle,\mathbb{Z}) \cong \mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z},\mathbb{Z})$. But then, since the projective resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

leads to the exact sequence

$$\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}) = 0 \longrightarrow \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \xrightarrow{\cdot n} \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \longrightarrow \mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}) \longrightarrow 0$$

we get $\mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$, which is a contradiction. □

Therefore, when one tackles Whitehead's problem there is no loss of generality in assuming A to be torsion-free. In what follows, we are going to study the torsion-free rank of $\mathrm{Ext}^1(A,\mathbb{Z})$, which will lead to the conclusion that all countable W-group are free.

Remark 4.2. Saharon Shelah [6] proved that, for groups whose cardinality is the smallest uncountable, Whitehead's Problem is undecidable. Indeed, in this context, both the affirmative and negative answers to Whitehead's problem are consistent with the set of ZFC axioms. That is to say, neither the affirmative nor the negative statement implies a contradiction within ZFC.

5 The torsion-free rank of $\mathrm{Ext}^1(A,\mathbb{Z})$

5.1 Preparatory Lemmas

Before discussing the ranks of $\mathrm{Ext}^1(A,\mathbb{Z})$, we prove some lemmas we need.

Lemma 5.1. *If A is torsion-free then*

$$|A| \leq \max\{r_0(A), \aleph_0\}.$$

Proof. Let M be an independent maximal system for A. Since A is torsion-free, every element of M has infinite order. For $g \in A \setminus \{0\}$, the system $\{M, g\}$ is no longer independent, which means that there exist $n, n_1, \dots, n_k \in \mathbb{Z}$ and $a_1, \dots, a_k \in M$ such that

$$ng = \sum_{i=1}^{k} n_i a_i.$$

Assume $ng' = ng$, then $n(g'-g) = 0$, giving that $g' = g$ (since A is torsion-free). So one can injectively associate a tuple $\{n, n_1, \dots, n_k, a_1, \dots, a_k\}$ to each element of A. It follows that

$$\begin{aligned} |A| &\leq \left| \bigsqcup_{k\in\mathbb{N}} \mathbb{Z}^{k+1} \times M^k \right| \\ &= \sum_{k\in\mathbb{N}} |M| \cdot \aleph_0 = \max\{|M|, \aleph_0\}. \end{aligned}$$

□

This proof is outlined following how Fuchs shows that the definition of ranks is well-posed (see *Infinite Abelian Groups*, [2, Thm. 16.3]).

Lemma 5.2. *Let A and B be two groups. If there exists $f : A \longrightarrow B$ surjective map or $g : B \longrightarrow A$ injective map, then $r_0(A) \geq r_0(B)$.*

Proof. Let $f : A \longrightarrow B$ be a surjective map and let $M = \{b_j\}_{i\in J}$ be an independent system of B, maximal with respect to the property of containing only elements of infinite order. By taking the preimages of the elements we obtain a system $M' = \{a_j\}_{j\in J}$ such that $b_j = f(a_j)$ for all $j \in J$. If $n_1 a_{j_1} + \dots + n_k a_{j_k} = 0$, then $n_1 b_{j_1} + \dots + n_k b_{j_k} = 0$ and, by the independence of M, we have that $n_h b_{j_h} = 0$, giving that $n_h = 0$ for all $h = 1, \dots, k$, since $ord(b_{j_h})$ is not finite. Hence M' is an independent system containing only elements of infinite order. The proof of the second part of the assumption is analogous. □

Lemma 5.3. *Let A be a group whose torsion-free rank is an infinite cardinal number and let B be a subgroup such that $r_0(B) < r_0(A)$: then $r_0(A/B) = r_0(A)$.*

Proof. Pick a maximal independent system $\{a_j\}_{j\in r_0(B)}$ of element of B of infinite order and extend it to a maximal independent system $\{a_i\}_{i\in r_0(A)}$ of element of A of infinite order. Since $r_0(B) < r_0(A)$ there exists a subset S of $r_0(A) \setminus r_0(B)$ of cardinality equal to $r_0(A)$ such that $\overline{a_i} \neq \overline{a_j}$ in A/B for all $i, j \in S$. Let $\{i_1, \dots, i_k\}$ be indices in S, and let $n_1, \dots, n_k$ be integer numbers. Assume

$$n_1 \cdot \overline{a_{i_1}} + \dots + n_k \cdot \overline{a_{i_k}} = \overline{0}.$$

Then $n_1 \cdot a_{i_1} + \dots + n_k \cdot a_{i_k} \in B$. This means that the element $n_1 \cdot a_{i_1} + \dots + n_k \cdot a_{i_k}$ depends on $\{a_j\}_{j\in r_0(B)}$. Since $\{a_{i_1}, \dots, a_{i_k}\} \cup \{a_j\}_{j\in r_0(B)}$ is an independent system, it follows that $n_1 \cdot a_{i_1} + \dots + n_k \cdot a_{i_k} = 0$. Hence the integers n_h are all 0 for all $h \in \{1, \dots, k\}$, therefore $\{\overline{a_i}\}_{i\in S}$ is an independent system of element of infinite order for A/B. By Lemma 5.2, we have $r_0(A/B) \leq r_0(A)$, which implies that $r_0(A) = |S| \leq r_0(A/B) \leq r_0(A)$. □

Lemma 5.4. *If A is a torsion-free group of rank 1 then it is isomorphic to a subgroup of $\mathbb{Q}$.*

Proof. Fix a maximal independent system, $\{a\}$: then for all $g \in A \setminus \{0\}$ there exists a least $m_g \in \mathbb{N}^*$ such that $m_g g = n_g a$, with $n_g \in \mathbb{N}^*$ (observe that n_g is unique). Consider the function $f : A \longrightarrow \mathbb{Q}$ that maps a into 1, 0 into 0 and g into $\frac{n_g}{m_g}$: obviously f is injective, and we claim that it is also a homomorphism: If g and h are respectively mapped into $\frac{n_g}{m_g}$ and $\frac{n_h}{m_h}$, then

$$\begin{aligned} (n_g\, m_h + n_h\, m_g) \cdot a &= m_h n_g \cdot a + m_g n_h \cdot a \\ &= m_h m_g \cdot g + m_g m_h \cdot h \\ &= m_g m_h \cdot (g + h). \end{aligned}$$

If $g + h$ is mapped into $\frac{k}{l}$ then $l(g + h) = ka$ and, multiplying by $m_g m_h$,

$$k m_g m_h \cdot a = l m_g m_h \cdot (g + h) = l(n_g\, m_h + n_h\, m_g) \cdot a.$$

Since A is torsion-free,

$$l(n_g\, m_h + n_h\, m_g) = k m_g m_h.$$

Thus

$$\frac{n_g}{m_g} + \frac{n_h}{m_h} = \frac{(n_g\, m_h + n_h\, m_g)}{m_g m_h} = \frac{k}{l},$$

as was to be shown. □

5.2 Base Case: A of torsion-free rank 1

Now we are ready to study the torsion-free rank of $\mathrm{Ext}^1(A, \mathbb{Z})$.

Lemma 5.5. *If A is a countable torsion-free abelian group then $r_0(\mathrm{Ext}^1(A, \mathbb{Z})) \leq 2^{\aleph_0}$. If in addition A is free, then $\mathrm{Ext}^1(A, \mathbb{Z}) = 0$.*

Proof. Consider the injective resolution of $\mathbb{Z}$

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

and apply the functor $\mathrm{Hom}(A, -)$ to it. We obtain the long exact sequence in cohomology

$$0 \longrightarrow \mathrm{Hom}(A,\mathbb{Z}) \longrightarrow \mathrm{Hom}(A,\mathbb{Q}) \xrightarrow{\pi_*} \mathrm{Hom}(A,\mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Ext}^1(A,\mathbb{Z}) \longrightarrow 0$$

given that $\mathrm{Ext}^1(A, \mathbb{Q}) = 0$ (being $\mathbb{Q}$ a divisible group). Therefore we have

$$\mathrm{Ext}^1(A, \mathbb{Z}) \cong \frac{\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})}{\pi_*(\mathrm{Hom}(A, \mathbb{Q}))}$$

and this implies that $|\mathrm{Ext}^1(A,\mathbb{Z})| \le |\mathrm{Hom}(A,\mathbb{Q}/\mathbb{Z})| \le {\aleph_0}^{\aleph_0}$. Hence $r_0(\mathrm{Ext}^1(A,\mathbb{Z})) \le 2^{\aleph_0}$.

If in addition A is free, as observed at the beginning of Section 4, then $\mathrm{Ext}^1(A,\mathbb{Z})$ is trivial. □

To proceed in the analysis we need the following:

Proposition 5.6. *Let A be a non-free subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$: then there exists a prime number p such that $1/p^k \in A$ for infinitely many k or there are infinitely many primes q such that $1/q \in A$.*

Remark 5.7. Before getting into the proof, recall that for relatively prime numbers m and n, we have that $m/n \in A$ if and only if $1/n \in A$. Indeed, by Bézout identity, you can pick integers s and t such that $sm + tn = 1$, or equivalently $s \cdot (m/n) + t = 1/n$.

If $1/m$ also belongs to A, then $1/m + 1/n = (m+n)/mn$ is an element of A too and so is $1/mn$, since $m+n$ and mn are relatively primes. It follows that, for relatively prime numbers m and n, we have $1/mn \in A$ if and only if both $1/m$ and $1/n$ belong to A.

Proof. Let us suppose that for each prime p there exists a maximum power of p, p^n, such that $1/p^n \in A$ and that there are finitely many primes $\{p_1, \dots, p_k\}$ such that $1/p_j \in A$ for any $j = 1, \dots, k$. Let n_j be the maximum exponent of p_j such that $1/p_j^{n_j} \in A$ for each $j = 1, \dots, k$. Since $p_j^{n_j}, p_i^{n_i}$ are relatively prime for each $i \ne j$, by Remark 5.7

$$y = \frac{1}{p_1^{n_1} \cdot \ldots \cdot p_k^{n_k}} \in A.$$

Pick now an element h in the complement $A \setminus \langle y \rangle$ (it exists since $\langle y \rangle \subsetneq A$, otherwise A would be free). Assume $h = s/r$ with s, r relatively prime. By Remark 5.7 we get $1/r \in A$ but $1/r \notin \langle y \rangle$ still. Let $r = q_1^{m_1} \cdot \ldots \cdot q_s^{m_s}$ be the factorization in prime numbers q_i's. The we have

$$\frac{1}{q_i^{m_i}} = \left(\prod_{l \ne i} q_l^{m_l} \right) / r \in A$$

for all $i = 1, \dots, s$. Since $\{p_1, \dots, p_k\}$ exhausts the set of primes p such that $1/p \in A$ we have that $q_i \in \{p_1, \dots, p_k\}$ for all $i = 1, \dots, s$. Thus (modulo a rearrangement) $q_i = p_i$ for all $i = 1, \dots, s$ and so $m_i \le n_i$ for all $i = 1, \dots, s$. Therefore $1/r = \left(\prod_{i=1}^{s} p_i^{n_i - m_i} \right) y \in \langle y \rangle$, which is a contradiction. □

Theorem 5.8. *Let A be a non-free subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$: then the torsion-free rank of $\mathrm{Ext}^1(A/\mathbb{Z},\mathbb{Z})$ is greater or equal to $2^{\aleph_0}$.*

Proof. Consider the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$: applying the functor $\mathrm{Hom}(A/\mathbb{Z}, -)$, one obtains

$$\mathrm{Ext}^1(A/\mathbb{Z},\mathbb{Z}) \cong \frac{\mathrm{Hom}(A/\mathbb{Z},\mathbb{Q}/\mathbb{Z})}{\pi_* \mathrm{Hom}(A/\mathbb{Z},\mathbb{Q})} \cong \mathrm{Hom}(A/\mathbb{Z},\mathbb{Q}/\mathbb{Z}),$$

because $\mathrm{Hom}(A/\mathbb{Z},\mathbb{Q}) = 0$. Moreover, since the quotient $A/\mathbb{Z}$ is a torsion group, by Lemma 2.2 we have

$$\begin{aligned} \mathrm{Ext}^1(A/\mathbb{Z},\mathbb{Z}) &\cong \mathrm{Hom}(A/\mathbb{Z},\mathbb{Q}/\mathbb{Z}) \\ &\cong \prod_p \mathrm{Hom}(T_p(A/\mathbb{Z}), \mathbb{Z}(p^\infty)). \end{aligned} \quad (5.1)$$

By Proposition 5.6, there are two cases to handle.

<u>First Case:</u> *There is a prime number p such that $1/p^k \in A$ for infinitely many k. In particular, $1/p^k \in A$ for all $k \in \mathbb{N}^*$.*

Hence there exists a subgroup of $T_p(A/\mathbb{Z})$ isomorphic to the Prüfer p-group. Since the Prüfer p-group is divisible (so injective), the contravariant functor $\mathrm{Hom}(-, \mathbb{Z}(p^\infty))$ is exact and the restriction to the subgroup $\mathbb{Z}(p^\infty)$ gives a surjection of abelian groups

$$\mathrm{Hom}(T_p(A/\mathbb{Z}), \mathbb{Z}(p^\infty)) \longrightarrow End(\mathbb{Z}(p^\infty)) \longrightarrow 0.$$

By Theorem 2.1, the group $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}(p^\infty))$ is isomorphic to $\mathbb{Z}_p$, which is a torsion-free group and of cardinality equal to $2^{\aleph_0}$. Therefore by Lemma 5.1, we have that

$$|\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}(p^\infty))| \le \max\{r_0(\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}(p^\infty))), \aleph_0\}$$

which implies that $2^{\aleph_0} \le r_0(\,\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}(p^\infty))\,)$ and gives the inequality needed to infer the equality

$$r_0(\,\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}(p^\infty))\,) = 2^{\aleph_0}.$$

Thus, by virtue of Lemma 5.2,

$$r_0(\,\mathrm{Hom}(T_p(A/\mathbb{Z}), \mathbb{Z}(p^\infty))\,) \ge 2^{\aleph_0}$$

and by equation (5.1) so is the torsion-free rank $r_0(\mathrm{Ext}^1(A/\mathbb{Z},\mathbb{Z}))$.

<u>Second Case:</u> *There are infinitely many primes $\{p_n\}_{n \in \mathbb{N}}$ such that $1/p_n \in A$.*

For any prime such that $1/p_n$ belongs to A there is a subgroup isomorphic to $\mathbb{Z}/p_n\mathbb{Z}$ in $T_{p_n}(A/\mathbb{Z})$ and so we have an injective map

$$\bigoplus_{n \in \mathbb{N}} \mathbb{Z}/p_n\mathbb{Z} \hookrightarrow \bigoplus_{n \in \mathbb{N}} T_{p_n}(A/\mathbb{Z}) \hookrightarrow T(A/\mathbb{Z}) = A/\mathbb{Z}.$$

The functor $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ is exact, since the group $\mathbb{Q}/\mathbb{Z}$ is injective, so when applied to the previous map gives a surjection

$$\mathrm{Hom}(A/\mathbb{Z},\mathbb{Q}/\mathbb{Z}) \twoheadrightarrow \mathrm{Hom}(\bigoplus_{n \in \mathbb{N}} \mathbb{Z}/p_n\mathbb{Z}, \mathbb{Q}/\mathbb{Z}).$$

By Lemma 5.2, we have then

$$r_0(\mathrm{Hom}(A/\mathbb{Z},\mathbb{Q}/\mathbb{Z})) \ge r_0(\,Hom_{\mathbb{Z}}\Big(\bigoplus_{n \in \mathbb{N}} \mathbb{Z}/p_n\mathbb{Z}, \mathbb{Q}/\mathbb{Z}\Big)\,).$$

By Lemma 2.2 we have the isomorphism

$$Hom_{\mathbb{Z}}\Big(\bigoplus_{n\in\mathbb{N}}\mathbb{Z}/p_n\mathbb{Z},\mathbb{Q}/\mathbb{Z}\Big)\cong\prod_{n\in\mathbb{N}}\mathrm{Hom}(\mathbb{Z}/p_n\mathbb{Z},\mathbb{Z}(p_n^\infty)).$$

Since a generator of $\mathbb{Z}/p_n\mathbb{Z}$ must go into an element whose order divides p_n, the above group

$$\mathrm{Hom}(\mathbb{Z}/p_n\mathbb{Z},\mathbb{Z}(p_n^\infty))$$

is actually $\mathrm{Hom}(\mathbb{Z}/p_n\mathbb{Z},\mathbb{Z}/p_n\mathbb{Z})$, which is, in turn, isomorphic to $\mathbb{Z}/p_n\mathbb{Z}$. Therefore we get that

$$r_0(\mathrm{Hom}(A/\mathbb{Z},\mathbb{Q}/\mathbb{Z}))\geq r_0(\prod_{n\in\mathbb{N}}\mathbb{Z}/p_n\mathbb{Z}).$$

Now let us give a partition of $\mathbb{N}$ into $\aleph_0$ sets $\{I_n\}_{n\in\mathbb{N}}$, each of them of cardinality $\aleph_0$: then we can write

$$\prod_{n\in\mathbb{N}}\mathbb{Z}/p_n\mathbb{Z}=\prod_{n\in\mathbb{N}}\Big(\prod_{p\in I_n}\mathbb{Z}/p\mathbb{Z}\Big).$$

Note that each $\prod_{p\in I_n}\mathbb{Z}/p\mathbb{Z}$ has at least an element a_n of infinite order. Therefore the subgroup $\prod_{p\in I_n}\langle a_n\rangle$ is a torsion-free group whose cardinality is equal to $2^{\aleph_0}$. By Lemma 5.1 its torsion-free rank is then equal to $2^{\aleph_0}$. This yields that the torsion-free rank of $\prod_{n\in\mathbb{N}}\mathbb{Z}/p_n\mathbb{Z}$ is also $2^{\aleph_0}$. Therefore also in this second case we get the desired inequality

$$\begin{aligned} r_0(\mathrm{Ext}^1(A/\mathbb{Z},\mathbb{Z})) &= r_0(\mathrm{Hom}(A/\mathbb{Z},\mathbb{Q}/\mathbb{Z}) \\ &\geq r_0(\prod_{n\in\mathbb{N}}\mathbb{Z}/p_n\mathbb{Z})=2^{\aleph_0}. \end{aligned}$$

□

Theorem 5.9. *Let A be a countable torsion-free group of torsion-free rank equal to 1. If A is free, then $Ext^1(A,\mathbb{Z})$ is trivial. Otherwise $r_0(Ext^1(A,\mathbb{Z}))=2^{\aleph_0}$.*

Proof. By Lemma 5.5, it is enough to show that if A is not free, then $r_0(\mathrm{Ext}^1(A,\mathbb{Z}))\geq 2^{\aleph_0}$. Let us apply the functor $\mathrm{Hom}(-,\mathbb{Z})$ to the exact sequence

$$0\longrightarrow\mathbb{Z}\longrightarrow A\longrightarrow A/\mathbb{Z}\longrightarrow 0;$$

we obtain the exact sequence

$$\mathrm{Hom}(\mathbb{Z},\mathbb{Z})\stackrel{\delta}{\longrightarrow}\mathrm{Ext}^1(A/\mathbb{Z},\mathbb{Z})\longrightarrow\mathrm{Ext}^1(A,\mathbb{Z})\longrightarrow 0=\mathrm{Ext}^1(\mathbb{Z},\mathbb{Z})$$

In particular we can write

$$\mathrm{Ext}^1(A,\mathbb{Z})\cong\frac{\mathrm{Ext}^1(A/\mathbb{Z},\mathbb{Z})}{\delta(\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}))}.$$

The image $\delta(\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}))$ is a subgroup of $\mathrm{Ext}^1(A/\mathbb{Z},\mathbb{Z})$ isomorphic to a quotient of $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z})$. Since $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z})$ is isomorphic to $\mathbb{Z}$, by Lemma 5.2 the torsion-free rank of $\delta(\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}))$ is at most 1.

Now, fix a maximal independent system $\{a\}$ of A: by Lemma 5.4, we can suppose that A is a subgroup of $\mathbb{Q}$ and that a is equal to 1, implying that $\mathbb{Z}\subseteq A$. Therefore A falls into the assumption of Theorem 5.8 and so the torsion free rank $r_0(\mathrm{Ext}^1(A/\mathbb{Z},\mathbb{Z}))\geq 2^{\aleph_0}$. Thus by Lemma 5.3 we can conclude that the torsion-free rank of $\mathrm{Ext}^1(A,\mathbb{Z})$ is greater than $2^{\aleph_0}$. □

5.3 Chase's and Stein's Theorems

We now merge the previous proof with the one contained in the book *Almost Free Modules* [4, Thm. XII.4.1] in order to present the result about the torsion-free rank of $\mathrm{Ext}^1(A,\mathbb{Z})$.

Theorem 5.10 (Chase's Characterization)**.** *Let A be a countable torsion-free group. If A is free, then $Ext^1(A,\mathbb{Z})$ is trivial. Otherwise $r_0(Ext^1(A,\mathbb{Z}))=2^{\aleph_0}$.*

Proof. By Lemma 5.5, we know already that

$$r_0(\mathrm{Ext}^1(A,\mathbb{Z}))\leq 2^{\aleph_0}.$$

Now we prove the converse inequality without the further assumption of $r_0(A)$ equal to 1. We split the proof in two cases according to whether $r_0(A)$ is finite or not.

<u>Case 1</u>: If A is not free and of finite rank n, proceed by induction on n. For n equal to 1 we have already done. So let A be with torsion-free rank equal to $n>1$. By Theorem 2.7 we can assume that A is indecomposable (i.e. a non-trivial group that cannot be expressed as direct sum of two subgroups). Let $M=\{a_1,\dots,a_n\}$ be a maximal independent system for A. Define the subgroup

$$B=\{a\in A\setminus\{0\}: a\text{ depends on }\{a_1,\dots,a_{n-1}\}\}\cup\{0\},$$

which is a countable torsion-free group of rank $n-1$.

Since

$$A=\{a\in A\setminus\{0\}: a\text{ depends on }\{a_1,\dots,a_n\}\}\cup\{0\},$$

A/B is a countable torsion-free group of rank 1. It cannot be free, else it would be projective, yielding that $A\cong B\bigoplus A/B$ is free, contrary to our assumptions. Hence A/B is not free and $r_0(\mathrm{Ext}^1(A/B,\mathbb{Z}))=2^{\aleph_0}$.

Now apply $\mathrm{Hom}(-,\mathbb{Z})$ to the short exact sequence

$$0\longrightarrow B\longrightarrow A\longrightarrow A/B\to 0$$

and get the long exact one

$$0\longrightarrow\mathrm{Hom}(A/B,\mathbb{Z})\longrightarrow\mathrm{Hom}(A,\mathbb{Z})\longrightarrow\mathrm{Hom}(B,\mathbb{Z})\stackrel{\delta}{\longrightarrow}\mathrm{Ext}^1(A/B,\mathbb{Z})\longrightarrow\mathrm{Ext}^1(A,\mathbb{Z})\longrightarrow\mathrm{Ext}^1(B,\mathbb{Z})\longrightarrow 0.$$

By exactness we have

$$0\longrightarrow\frac{\mathrm{Ext}^1(A/B,\mathbb{Z})}{\delta(\mathrm{Hom}(B,\mathbb{Z}))}\longrightarrow\mathrm{Ext}^1(A,\mathbb{Z}).$$

Since the rank of B is finite, the group $\mathrm{Hom}(B,\mathbb{Z})$ is at most countable and thus $|\delta(\mathrm{Hom}(B,\mathbb{Z}))| \leq \aleph_0$.

Since $\aleph_0 < 2^{\aleph_0}$, by Lemma 5.3 the torsion-free rank of the quotient $\dfrac{\mathrm{Ext}^1(A/B,\mathbb{Z})}{\delta(\mathrm{Hom}(B,\mathbb{Z}))}$ remains $2^{\aleph_0}$. By Lemma 5.2, the torsion-free rank $r_0(\mathrm{Ext}^1(A,\mathbb{Z}))$ is $2^{\aleph_0}$.

Case 2: If A is not free and of infinite rank by Pontryagin's criterion there is a subgroup $B \subseteq A$ of finite rank which is not free. Now consider the short exact sequence

$$0 \longrightarrow B \longrightarrow A \longrightarrow A/B \longrightarrow 0$$

and apply $\mathrm{Hom}(-,\mathbb{Z})$. The result is:

$$\begin{array}{l} 0 \longrightarrow \mathrm{Hom}(A/B,\mathbb{Z}) \longrightarrow \mathrm{Hom}(A,\mathbb{Z}) \longrightarrow \mathrm{Hom}(B,\mathbb{Z}) \\ \longrightarrow \mathrm{Ext}^1(A/B,\mathbb{Z}) \longrightarrow \mathrm{Ext}^1(A,\mathbb{Z}) \longrightarrow \mathrm{Ext}^1(B,\mathbb{Z}) \longrightarrow 0. \end{array}$$

By Lemma 5.2 $r_0(\mathrm{Ext}^1(A,\mathbb{Z})) \geq r_0(\mathrm{Ext}^1(B,\mathbb{Z})) = 2^{\aleph_0}$ and therefore $r_0(\mathrm{Ext}^1(A,\mathbb{Z})) = 2^{\aleph_0}$. □

The solution for Whitehead's problem for countable groups is now an easy corollary:

Corollary 5.11 (Stein's Theorem)**.** *Let A be a countable group. Then $Ext^1(A,\mathbb{Z}) = 0$ if and only if A is free.*

6 Concluding remarks

For completeness, we would like to conclude with a couple of further remarks about the analogue of Theorem 5.10 for p-ranks of $\mathrm{Ext}^1(A,\mathbb{Z})$ and the structure of the group itself, when A is countable and torsion-free.

Lemma 6.1. *Let A be a torsion-free group: then $Ext^1(A,\mathbb{Z})$ is injective.*

Proof. As seen in the proof of Lemma 5.5, we have

$$\mathrm{Ext}^1(A,\mathbb{Z}) = \frac{\mathrm{Hom}(A,\mathbb{Q}/\mathbb{Z}))}{\pi_*(\mathrm{Hom}(A,\mathbb{Q}))}.$$

In particular if $\mathrm{Hom}(A,\mathbb{Q}/\mathbb{Z})$ is injective so will be $\mathrm{Ext}^1(A,\mathbb{Z})$.

Consider the multiplication $A \xrightarrow{\cdot n} A$ by a non-zero integer n: it is injective because A is torsion-free. By injectivity of the group $\mathbb{Q}/\mathbb{Z}$, any map $\phi : A \longrightarrow \mathbb{Q}/\mathbb{Z}$ can be written as the composite of a map ψ with $\cdot n$ as in the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{\cdot n} & A \\ & & {\scriptstyle\phi}\downarrow & \swarrow{\scriptstyle\psi} & \\ & & \mathbb{Q}/\mathbb{Z}. & & \end{array}$$

This is equivalent to say that $\mathrm{Hom}(A,\mathbb{Q}/\mathbb{Z})$ is divisible. □

As touched on before, Chase also computed the p-ranks of $\mathrm{Ext}^1(A,\mathbb{Z})$ for A countable and torsion-free. The analogue of Chase's Characterization of the torsion-free rank (Theorem 5.10) is the following, whose proof is out of the scope of this article but it can be found in the book *Almost Free Modules* (more precisely [4, XII, Thm. 4.7]).

Theorem 6.2 (Chase's Characterization for p-ranks)**.** *If A is a countable torsion-free group, then for any prime p, the rank $r_p(Ext^1(A,\mathbb{Z}))$ is either finite or $2^{\aleph_0}$.*

Therefore, if we apply Theorem 3.5 to the divisible group $\mathrm{Ext}^1(A,\mathbb{Z})$, it provides a complete characterization of $\mathrm{Ext}^1(A,\mathbb{Z})$ in terms of its ranks.

Corollary 6.3. *If A is countable and torsion-free, then*

$$Ext^1(A,\mathbb{Z}) \cong \mathbb{Q}^{\oplus r_0(Ext^1(A,\mathbb{Z}))} \oplus \bigoplus_p \mathbb{Z}(p^\infty)^{\oplus r_p(Ext^1(A,\mathbb{Z}))}$$

where all the ranks are either finite or $2^{\aleph_0}$.

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