UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

Department of Electrical and Computer Engineering

ECE 486: Control Systems

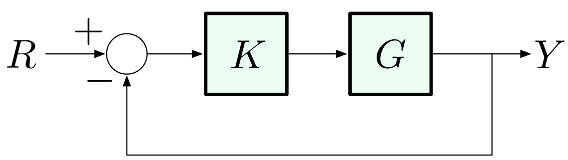
Homework 4 Solutions

Fall 2018

Problem 1

Consider the following feedback system, where K is a constant gain and

$$G(s) = \frac{1}{s^3 + 2s^2 + 2s + 1} :$$



The transfer function of the system can be written as:

$$\begin{aligned}\frac{Y}{R} &= \frac{KG}{1+KG}\\ &= \frac{\frac{K}{s^3+2s^2+2s+1}}{1+\frac{K}{s^3+2s^2+2s+1}}\\ &= \frac{K}{s^3+2s^2+2s+1+K}\end{aligned}$$

In order to satisfy the necessary condition, all coefficients should be positive.

$$\therefore K+1>0 \quad \Rightarrow \quad K>-1$$

Use the Routh-Hurwitz criterion to check for other conditions:

s^3	$: 1$	2
s^2	$: 2$	$K+1$
s^1	$: b_1$	0
s^0	$: c_1$	

$$
\begin{aligned}
b_1 &= -\frac{1}{2}\begin{vmatrix} 1 & 2 \\ 2 & K+1 \end{vmatrix} = -\frac{1}{2}(K+1-4) \\
&= -\frac{1}{2}(K-3) \\
c_1 &= -\frac{1}{-\frac{1}{2}(K-3)}\begin{vmatrix} 2 & K+1 \\ -\frac{1}{2}(K-3) & 0 \end{vmatrix} \\
&= \frac{1}{\frac{1}{2}(K-3)}(\frac{1}{2}(K-3)(K+1)) \\
&= K+1
\end{aligned}
$$

From the test, we can see that there are two conditions:

$$-\frac{1}{2}(K-3) > 0 \Rightarrow K < 3$$

and

$$K+1 > 0 \Rightarrow K > -1$$

$\therefore -1 < K < 3$ for a stable closed loop system.

Problem 2

Consider the same feedback configuration as in Problem 1, but now with $K(s)$ and $G(s)$ unknown transfer functions. Suppose that we know that the transfer function from R to Y is

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

for some parameters $\zeta > 0$ and $\omega_n > 0$.

(i) Based on this information, find the forward gain $K(s)G(s)$.

$$
\begin{aligned}
\frac{Y}{R} &= \frac{K(s)G(s)}{1+K(s)G(s)} \\
&= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2} \\
&= \frac{\omega_n^2}{s(s+2\zeta\omega_n) + \omega_n^2} \\
&= \frac{\frac{\omega_n^2}{s(s+2\zeta\omega_n)}}{1 + \frac{\omega_n^2}{s(s+2\zeta\omega_n)}} \\
\therefore K(s)G(s) &= \frac{\omega_n^2}{s(s+2\zeta\omega_n)}
\end{aligned}
$$

(ii) Determine the system type and discuss what it implies about the closed-loop system's steady-state tracking capability.

The system has one pole at the origin. Therefore, the system is Type I system.

$$k_p = \lim_{s\to 0} K(s)G(s) = \lim_{s\to 0} \frac{\omega_n^2}{s(s+2\zeta\omega_n)} = \infty \qquad \Rightarrow \frac{1}{1+k_p} = 0$$

The system follows constant references (step) without error.

$$k_v = \lim_{s\to 0} sK(s)G(s) = \lim_{s\to 0} s\frac{\omega_n^2}{s(s+2\zeta\omega_n)} = \frac{\omega_n}{2\zeta} \qquad \Rightarrow \frac{1}{k_v} = \frac{2\zeta}{\omega_n}$$

The system follows ramp references with constant error $\frac{2\zeta}{\omega_n}$.

$$k_a = \lim_{s\to 0} s^2K(s)G(s) = \lim_{s\to 0} s^2\frac{\omega_n^2}{s(s+2\zeta\omega_n)} = 0 \qquad \Rightarrow \frac{1}{k_a} = \infty$$

The system cannot follow parabola references.

Problem 3

The *speed* y of a DC motor satisfies $\dot{y}+60y = 600v - 1500d$, where v is the armature voltage — the input to the system — and d is a load.

Suppose v is defined via the PI control law,

$$v = K_P e + K_I \int_0^t e(s)\,\mathrm{d}s.$$

where $e = r - y$, as usual, with r the reference speed.

(i) Define a disturbance process $\tilde{d}$ so that the model can be expressed in the 'input disturbance' form,

$$Y = G_p(V + \tilde{D})$$

where G_p is the plant transfer function.

Model: $\dot{y} + 60y = 600v - 1500d \Rightarrow sY(s) + 60Y(s) = 600V(s) - 1500D(s)$

$$\begin{aligned} Y(s) &= \frac{600}{s+60}V(s) - \frac{1500}{s+60}D(s) \\ &= \frac{600}{s+60}\left[V(s) + \left(\frac{-1500}{600}\right)D(s)\right] \end{aligned}$$

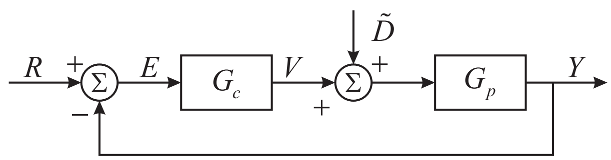
With $G_p(s) = \dfrac{600}{s+60}$ and $\tilde{D}(s) = \dfrac{-1500}{600}D(s) = -\dfrac{5}{2}D(s)$, we obtain

$$Y(s) = G_p(s)[V(s) + \tilde{D}(s)]$$

(ii) For this part, we consider PI compensation

$$v(t) = K_P e(t) + K_I \int_0^t e(s)ds$$
$$V(s) = K_P E(s) + \frac{K_I}{s} E(s)$$
$$= \left(K_P + \frac{K_I}{s} \right) E(s) = G_c(s)E(s)$$

Block diagram:



(iii) Ignore $\tilde{D}$

$$H_1(s) = \frac{Y(s)}{R(s)} = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)} = \frac{\frac{600}{s+60}\frac{K_P s + K_I}{s}}{1 + \frac{600}{s+60}\frac{K_P s + K_I}{s}}$$
$$= \frac{600(K_P s + K_I)}{s(s+60) + 600(K_P s + K_I)}$$

Note: $H_1(0) = \dfrac{600K_I}{600K_I} = 1$, as expected.

(iv) Ignore R

From Part (i), we found that $\tilde{D}(s) = -\frac{5}{2}D(s)$

$$H_2(s) = \frac{Y(s)}{D(s)} = -\frac{5}{2} \cdot \frac{Y(s)}{\tilde{D}(s)} = -\frac{5}{2}\left(\frac{G_p(s)}{1 + G_c(s)G_p(s)} \right)$$
$$= -\frac{5}{2}\left(\frac{\frac{600}{s+60}}{1 + \frac{600}{s+60}\frac{K_P s + K_I}{s}} \right)$$
$$= -\frac{5}{2}\left(\frac{600s}{s(s+60) + 600(K_P s + K_I)} \right)$$
$$= -\frac{1500s}{s(s+60) + 600(K_P s + K_I)}$$

Note: $H_2(0) = 0$, as expected.

(v) Compute values of K_P, K_I so that the following closed loop specifications hold for a step reference input: no more than 5% overshoot, $t_s^{5\%} = 1$ sec. and no undershoot (no zeros in RHP).

Take $d \equiv 0$.

Assume that poles are complex

We know that $t_s^{5\%} \approx \frac{3}{\sigma}$. Our desired $t_s^{5\%}$ is 1 sec.

$$\therefore \frac{3}{\sigma} = 1 \Rightarrow \sigma = \frac{3}{1} = 3$$

Recall the characteristic equation: $s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 2\sigma s + \omega_n^2$

From part (iii),

$$\begin{aligned}\frac{Y(s)}{R(s)} &= 600\frac{(K_P s + K_I)}{s(s+60) + (K_P s + K_I)} \\ &= 600\frac{K_P s + K_I}{s^2 + (60 + 600K_P)s + 600K_I}\end{aligned}$$

Therefore, this requires $2\sigma = (60 + 600K_P)$

$$2\sigma = 60 + 600K_P \Rightarrow \sigma = 30 + 300K_P \Rightarrow 3 = 30 + 300K_P \therefore K_P = -\frac{30-3}{300} = -\frac{9}{100}$$

Now, consider the overshoot spec, $M_p \leq 0.05$

$$\begin{aligned}M_p = e^{\frac{-\pi\cdot\zeta}{\sqrt{1-\zeta^2}}} &\leq 0.05 \\ \therefore 0.69 \leq \zeta &< 1\end{aligned}$$

Let $\zeta = 0.69$

We found earlier that

$$\sigma = \zeta\omega_n = 3$$

$$\therefore \omega_n = \frac{3}{\zeta} = \omega_n = \frac{3}{0.69} = 4.35$$

Back to our characteristic equation, this requires that $\omega_n^2 = 600K_I$

$$\omega_n^2 = 4.35^2 = 600K_I \Rightarrow K_I = \frac{18.92}{600} = 0.032$$

HOWEVER , consider the location of the zero, which is determined by $K_P s + K_I = 0$
If $K_P = -\frac{9}{100}$ and $K_I = 0.032$, then the zero is at $s = \frac{-K_I}{K_P} = \frac{0.032 * 100}{9} > 0$ which is in the RHP $\Rightarrow$ Undershoot $\Rightarrow$ Spec not satisfied.

Repeating the steps with $\zeta = 1$ (No overshoot) still didn't eliminate the undershoot. Therefore, the assumption that poles are complex is incorrect.

Now, assume that poles are real.

The desired characteristic polynomial is

$$
\begin{aligned}
(s+\sigma)(s+p) &= (s+3)(s+p) \\
&= s^2 + (p+3)s + 3p \\
&= s^2 + (60 + 600K_P)s + 600K_I
\end{aligned}
$$

Require $K_P, K_I > 0$ to avoid undershoot.

Try $K_P = \frac{1}{10}$, which gives $p + 3 = 60 + 600K_P = 120$ or $p = 120 - 3 = 117$.

The pole location is $-p$ which is much less than -4. Then,

$$3p = 300K_I \Rightarrow K_I = \frac{3(117)}{600} = 0.585$$

We now have a zero at $-\frac{k_I}{k_p} \cong -\frac{0.585}{0.1} = -5.85$, which in in the OLHP $\Rightarrow$ No longer have undershoot

Poles placed at {-3,-117}. Results in a zero in the LHP at -5.85. $\Rightarrow$ Perfect step response!

Problem 4

Set up the listed characteristic equations in the form suited to Evans' root-locus method, as described in Section 5.1. In each case, express $a(s)$, $b(s)$, and K in terms of the original parameters. Remember that the polynomials $a(s)$ and $b(s)$ must be monic, and the degree of $b(s)$ no greater than the degree of $a(s)$.

(i) To obtain $n \geq m$ (proper transfer function), we write

$$s + \frac{1}{\tau^2} = 0 \Rightarrow 1 + K\frac{1}{s} = 0, \quad K = \frac{1}{\tau^2}$$

(ii) As in (a), write

$$1 + K\frac{s}{(s+R)^3} = 0, \quad K = \frac{1}{T}$$

(iii) Now, this is impossible without changing the problem. Write $s' = s + R$, so that

$$\begin{aligned} (s+R)^3 + \frac{s}{T} &= (s')^3 + \frac{1}{T}(s' - R) \\ &= (s')^3 + \frac{1}{T}s' - K, \quad K = \frac{R}{T} \end{aligned}$$

This results in a negative root locus.

$$1 - K\frac{1}{(s')^3 + \frac{1}{T}s'} = 0$$