Systems Analysis and Control

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Lecture 14: Root Locus Continued

Overview

In this Lecture, you will learn:

Review: What happens at high gain?

- Angles of Departure

The Case of 90° Departure

- Calculating the center of asymptotes

Breaking off the Real Axis

- Break Points

What is the effect of small gain?

- Departure Angles

Pole locations change at high gain.

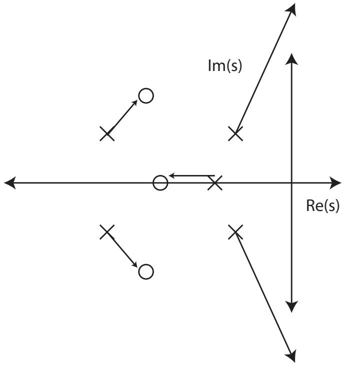
- Some poles stay small
- Some poles get large
  - Asymptotes depend on relative number of poles and zeros.

Small poles go to zeros.

Big poles leave on asymptotes:

Cases:

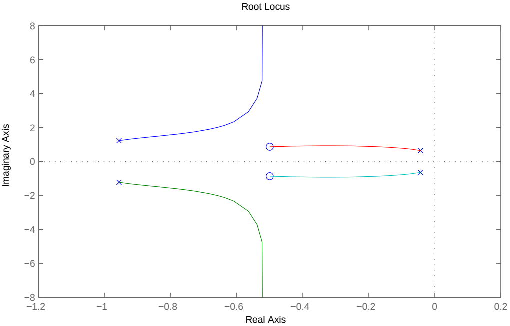
- $n - m = 0$ - No Asymptotes
- $n - m = 1$ - Asymptote at $180°$
- $n - m = 2$ - Asymptotes at $\pm 90°$
- $n - m = 3$ - Asymptotes at $180°$, $\pm 60°$
- $n - m = 4$ - Asymptotes at $\pm 45°$ and $\pm 135°$



Recall the suspension system:
$G(s) = \frac{s^2+s+1}{s^4+2s^3+3s^2+1s+1}$.

Count: 2 zeros, 4 poles.

$$n - m = 2$$



$$\angle_\infty = -90^\circ, -270^\circ$$

2 vertical asymptotes at 90° and 270°.

Poles **MAY** destabilize at large gain. But will they???

- Why these poles?

The Asymptotic Center

Recall

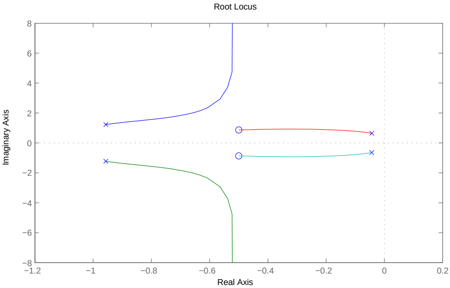
- $m = \#$ of zeroes
- $n = \#$ of poles

Problem 1: When $n - m = 2$.

- Is high gain destabilizing?

Problem 2: When $n - m \geq 2$.

- Which poles get big?



Definition 1.

The **Center of Asymptotes** is where all asymptotes meet.

The center of asymptotes is only for the *big* poles on the root locus.

- The center of asymptotes is the average of these points as $k \to \infty$.

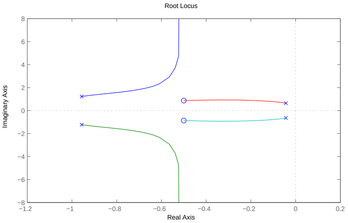
$$center = \frac{\sum q_{i_{BIG}}}{\#_{i_{BIG}}}$$

Calculating the Asymptotic Center

$$center = \frac{\sum q_{i_{BIG}}}{\#_{i_{BIG}}}$$

Denote

- q_i are the CLOSED-LOOP poles
  - q_i are roots of $d(s) + kn(s)$
- z_i are the zeros (open and closed loop)
  - z_i are roots of $n(s)$
- p_i are the OPEN-LOOP poles
  - p_i are roots of $d(s)$



Calculating $\#_{i_{BIG}}$ is easy!

- Small poles go to zeroes
- Big poles form asymptotes

$$\#_{i_{BIG}} = n - m = \#OL\ poles - \#OL\ zeroes$$

Real Problem: How to calculate

$$\sum q_{i_{BIG}}?$$

Calculating the Asymptotic Center

Recall from Routh-Hurwitz: Let p_i be the roots of $d(s)$.

$$d(s) = s^n + a_1 s^{n-1} + \cdots + a_n = (s - p_1)(s - p_2)\cdots(s - p_n)$$

Observe what happens as we expand out the roots:

$$\begin{aligned}
&d(s) = (s - p_1)(s - p_2)(s - p_3)(s - p_4)\cdots(s - p_n) \\
&= (s^2 - (p_1 + p_2)s + p_1 p_2)(s - p_3)(s - p_4)\cdots(s - p_n) \\
&= (s^3 - (p_1 + p_2 + p_3)s^2 + (p_1 p_2 + p_2 p_3 + p_1 p_3)s - p_1 p_2 p_3)(s - p_4)\cdots(s - p_n) \\
&= \cdots \\
&= s^n - (p_1 + p_2 + \cdots + p_n)s^{n-1} + \cdots + (-1)^n p_1 p_2 \cdots p_n
\end{aligned}$$

The second coefficient is the negative sum of the roots

$$a_1 = -(p_1 + p_2 + \cdots + p_n) = -\sum p_i$$

Calculating the Asymptotic Center

Since $\frac{kG}{1+kG} = \frac{kn}{d+kn}$, $\sum q_i$ is the second coefficient of

$$d(s) + kn(s)$$

Only interested in the case when $n - m \geq 2$

- 90° asymptotes or more.

$$d(s) = s^n + a_1 s^{n-1} + \cdots$$
$$n(s) = s^m + \cdots$$

When $n - m = 2$,

$$d(s) + kn(s) = s^n + a_1 s^{n-1} + (a_2 + k)s^{n-2} + \cdots$$

Conclusion: Changing k doesn't change the second coefficient.

- Sum of poles doesn't change under feedback.

$$\sum p_i = \sum q_i = -a_1$$

This sum is the second coefficient of $d(s)$.

Calculating the Asymptotic Center

Recall we want to find

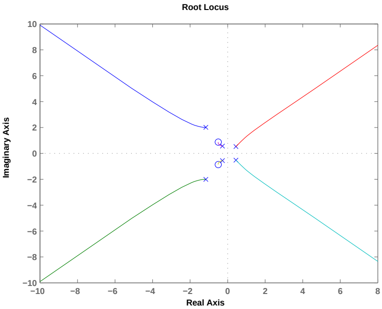
$$center = \frac{\sum q_{i_{BIG}}}{\#_{i_{BIG}}}$$

It is obvious that

$$\sum q_i = \sum q_{i_{BIG}} + \sum q_{i_{SMALL}} = -a_1$$

So that

$$\sum q_{i_{BIG}} = -a_1 - \sum q_{i_{SMALL}}$$



So how do we find $\sum q_{i_{SMALL}}$?

- As $k \to \infty$ small poles go to zeroes.

At high gain

$$\sum q_{i_{SMALL}} \cong \sum z_i$$

Calculating the Asymptotic Center

$$G(s) = \frac{n(s)}{d(s)}$$

The zeros, z_i are the roots of $n(s)$.

$$n(s) = s^m + b_1 s^{m-1} + \cdots = (s - z_1)\cdots(s - z_m)$$

As before

$$\sum z_i = -b_1$$

Finally

$$center = \frac{\sum q_{i_{BIG}}}{\#_{i_{BIG}}} = \frac{\sum q_i - \sum q_{i_{SMALL}}}{n-m} = \frac{\sum p_i - \sum z_i}{n-m} = \frac{b_1 - a_1}{n-m}$$

Where

- a_1 is the first coefficient of $d(s)$
- b_1 is the first coefficient of $n(s)$

Calculating the Asymptotic Center

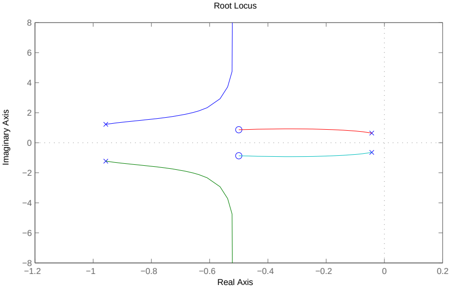
Example: Suspension System

$$G(s) = \frac{s^2 + s + 1}{s^4 + 2s^3 + 3s^2 + 1s + 1}$$

$$\begin{aligned}\#_{i_{BIG}} &= n - m \\ &= \#poles - \#zeroes \\ &= 2.\end{aligned}$$

Read off the coefficients

- $a_1 = 2$
- $b_1 = 1$



$$center = \frac{b_1 - a_1}{n - m} = \frac{1 - 2}{2} = -\frac{1}{2}$$

Conclusion: High gain is stable.

Calculating the Asymptotic Center

Example: Tweaked Suspension System

Look what happens if we change 2^{nd} coefficient in $n(s)$ from 1 to 3.

$$G(s) = \frac{s^2 + 3s + 1}{s^4 + 2s^3 + 3s^2 + 1s + 1}$$

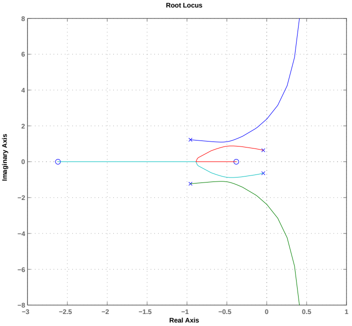
$$\#_{i_{BIG}} = n - m = \#poles - \#zeroes = 2$$

Read off the coefficients

- $a_1 = 2$
- $b_1 = 3$

Thus

$$center = \frac{b_1 - a_1}{n - m} = \frac{3 - 2}{2} = \frac{1}{2}$$



Now high gain is unstable.

Calculating the Asymptotic Center

Example: Suspension System with Integral Feedback

$$G(s) = \frac{s^2+s+1}{s^4+2s^3+3s^2+1s+1}\frac{1}{s}$$
$$= \frac{s^2+s+1}{s^5+2s^4+3s^3+1s^2+s}$$

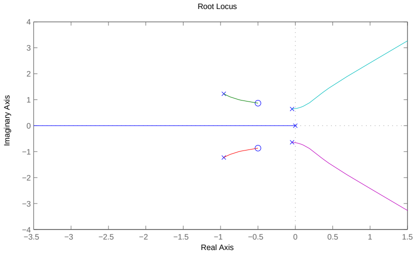
$\#_{i_{BIG}} = n - m =$
$\#poles - \#zeroes = 3.$

Again, we have the same coefficients

- $a_1 = 2$
- $b_1 = 1$

Thus

$$center = \frac{b_1 - a_1}{n-m} = \frac{1-2}{3} = -\frac{1}{3}$$



Calculating the Asymptotic Center

Another Example

$$G(s) = \frac{s^2+s+1}{s^6+2s^5+5s^4-s^3+2s^2+1}$$

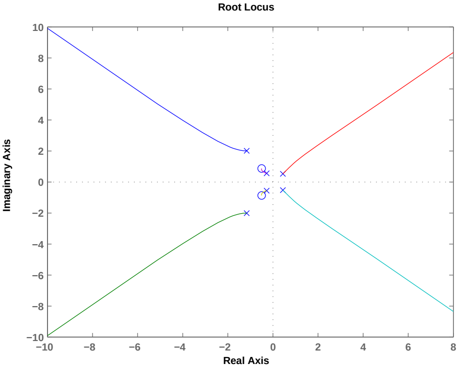
First, $\#_{i_{BIG}} = n - m = 4$.

Again, we have the same coefficients

- $a_1 = 2$
- $b_1 = 1$

Thus

$$center = \frac{b_1 - a_1}{n-m} = \frac{1-2}{4} = -\frac{1}{4}$$



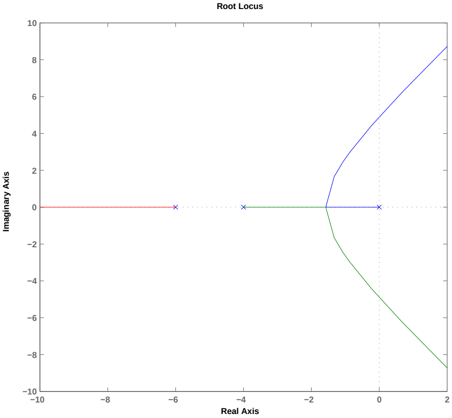
$$G(s) = \frac{1}{s(s+4)(s+6)}$$

First, $\#_{i_{BIG}} = n - m = 3$

This time, we can directly use poles and zeros

- No Zeroes
- $p_1 = 0$, $p_2 = -4$, $p_3 = -6$.

$$\sum p_i = -4 - 6 = -10$$

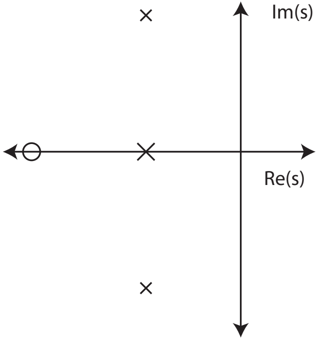


$$center = \frac{\sum p_i - \sum z_i}{n - m} = \frac{-10 - 0}{3} = -3.33\bar{3}$$

Calculating the Asymptotic Center

DIY Example

$$G(s) = \frac{s+2}{(s+1)(s^2+2s+2)}$$



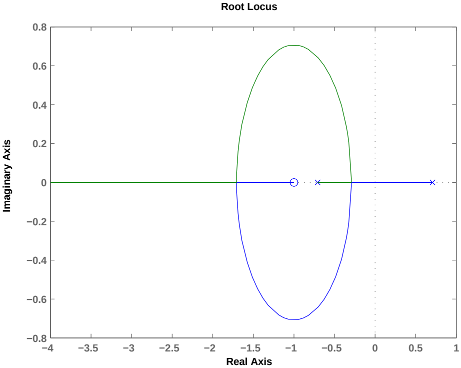
Break points

Recall the inverted pendulum with derivative feedback.

$$G(s) = \frac{1+s}{s^2 - \frac{1}{2}}$$

When do the poles become imaginary?

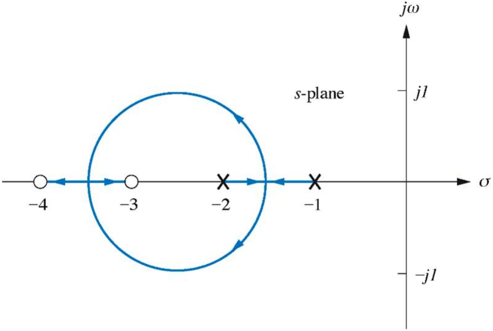
- Important for choosing optimal k.



Break points

Other Examples

$$G(s) = \frac{(s+3)(s+4)}{(s+1)(s+2)}$$



Break points

Recall for a point on the root locus

$$d(s) + kn(s) = 0$$

or for a point on the real axis: $s = a$

$$k(a) = -\frac{d(a)}{n(a)} = -\frac{1}{G(a)}$$

Idea: Use maximum principle to find the maximum and minimum of k on the real axis.

Definition 2.

The extrema of a continuous function of a real variable, $f(a)$, occur at the boundary or when

$$\frac{d}{da} f(a) = 0$$

Break points

To find the point when the root locus leaves the real axis, we calculate the extrema of

$$k(a) = -\frac{1}{G(a)}$$

We need to solve

$$\frac{d}{da}k(a) = 0$$

or

$$\frac{d}{da}k(a) = -\frac{d}{da}\frac{1}{G(a)} = \frac{d(a)}{n(a)^2}n'(a) - \frac{d'(a)}{n(a)} = \frac{d(a)n'(a) - d'(a)n(a)}{n(a)^2} = 0$$

Break Points occur at real-valued solutions of

$$d(a)n'(a) - d'(a)n(a) = 0$$

Break points

Numerical Example

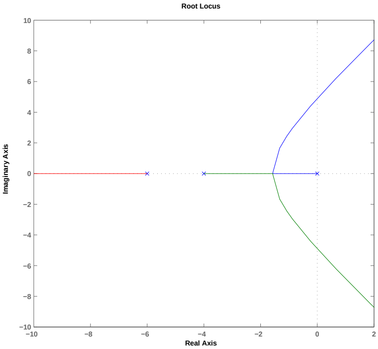
$$G(s) = \frac{1}{s(s+4)(s+6)} = \frac{1}{s^3 + 10s^2 + 24s}$$

Break points occur when

$$d(a)n'(a) - d'(a)n(a)$$
$$= 0 - (3a^2 + 20a + 24) = 0$$

which has roots

$$a_{1,2} = \frac{-20 \pm \sqrt{20^2 - 4 * 24 * 3}}{6}$$
$$\cong -5.1,\ -1.57$$



Break points

Numerical Example

$$G(s) = \frac{(s+3)(s+4)}{(s+1)(s+2)} = \frac{s^2+7s+12}{s^2+3s+2}$$

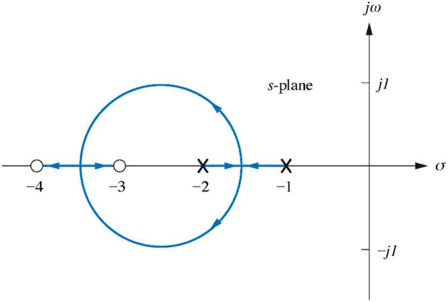
Break points occur when

$$\begin{aligned} d(a)n'(a) - d'(a)n(a) &= (a^2+3a+2)(2a+7) - (2a+3)(a^2+7a+12) \\ &= (a^2+3a+2)(2a+7) - (2a+3)(a^2+7a+12) \\ &= -2(2a^2+10a+11) = 0 \end{aligned}$$

Which has roots

$$a_{1,2} = -1.634,\ -3.366$$

Break points at -1.634 and -3.366.



Break points

Numerical Example

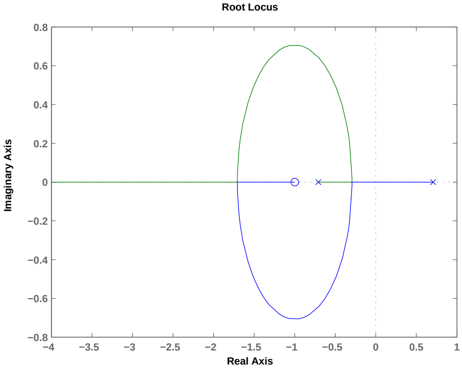
$$G(s) = \frac{1+s}{s^2 - \frac{1}{2}}$$

Break points occur when

$$\begin{aligned} & d(a)n'(a) - d'(a)n(a) \\ & \quad = (a^2 - .5)\cdot 1 - 2a\cdot(1+a) \\ & \quad = -(a^2 + 2a + .5) = 0 \end{aligned}$$

Which has roots

$$a_{1,2} = -.293,\ -1.707$$



Break points at $-.293$ and -1.707

Break points

Summary

Step 1: Root Locus starts at Open Loop Poles.
Step 2: At Large Gain, $k \to \infty$

- Small Poles go to zeroes
- Large Poles approach asymptotes
- Center at

$$\sigma = \frac{\sum p_i - \sum z_i}{n-m} = \frac{b_1 - a_1}{n-m}$$

Step 3: On real axis

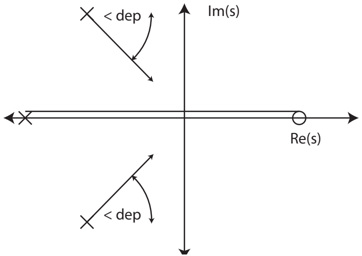
- When odd number of poles/zeroes to the right.
- Break points when

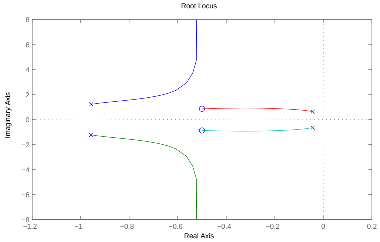
$$-\frac{d}{da}\frac{1}{G(a)} = 0 \qquad \textbf{or} \qquad d(a)n'(a) - d'(a)n(a) = 0$$

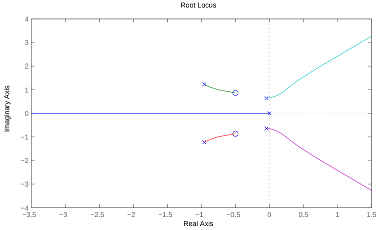
Departure Angle

The root locus starts at the poles.

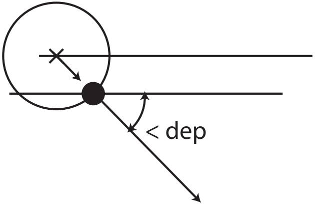
- What it the effect of small gain?
- Do the poles become more or less stable?

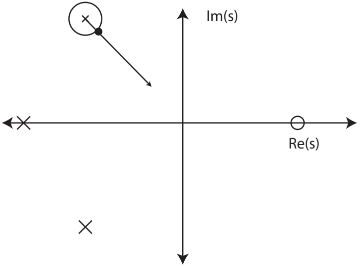






To find the departure angle, we look at a very small region around the departure point.





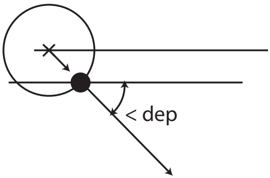
For a point to be on the root locus, we want phase of 180°.

$$\angle G(s) = \sum_{i=1}^{m} \angle(s - z_i) - \sum_{i=1}^{n} \angle(s - p_i) = 180^\circ$$

Departure Angle

If we make the point s extremely close to the pole p.

- The angle to other poles and zeros from s is the same as from p.
  - $\angle(s - z_i) \cong \angle(p - z_i)$ for all i
  - $\angle(s - p_i) \cong \angle(p - p_i)$ for all i
- The only difference is the phase from p itself.

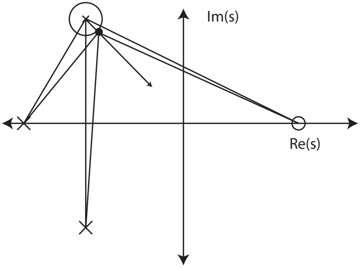


The phase due to p equals the departure angle, $\angle_{dep}$

$$\angle(s - p) = \angle_{dep}$$

The total phase is

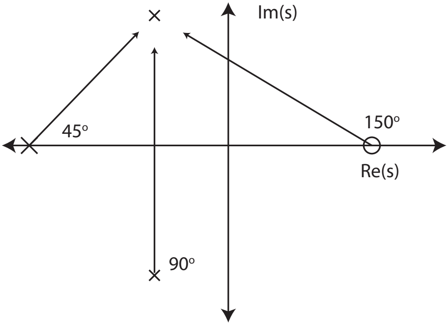
$$\angle G(s) = \angle G(p) - \angle_{dep} = 180^\circ$$



Thus the departure angle from pole p is

$$\angle_{dep} = \angle G(p) + 180^\circ$$

Therefore, to find the departure angle from pole p, just find the phase at p.



The phase at p is based on geometry.

$$\angle G(p) = 150^\circ - 90^\circ - 45^\circ = 15^\circ$$

So the departure angle is easy to calculate.

$$\angle_{dep} = \angle G(p) + 180^\circ = 195^\circ$$

Departure Angle

Numerical Examples

$$G(s) = \frac{s^2 + s + 1}{s^4 + 2s^3 + 3s^2 + 1s + 1}$$

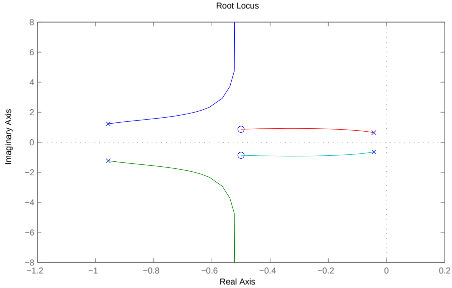
Poles at

- $p_{1,2} = -.957 \pm 1.23$
- $p_{3,4} = -.0433 \pm .641$

Zeroes at

- $z_{1,2} = -.5 \pm .866\imath$

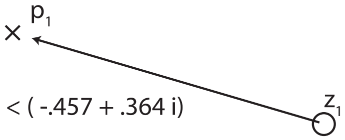
Problem:
Find departure angle at $p_1 = -.957 + 1.23$.



$$\angle_{dep} = 180^\circ + \angle(p_1 - z_1) + \angle(p_1 - z_2) - \angle(p_1 - p_2) - \angle(p_1 - p_3) - \angle(p_1 - p_4)$$

The difficulty is calculating the phase.

$$\begin{aligned}\angle(p_1 - z_1) &= \angle(-.957 + 1.23\imath + .5 - .866\imath)\\ &= \angle(-.457 + .364\imath)\\ &= \tan^{-1}\left(\frac{.364}{-.457}\right)\\ &= 141.46^\circ\end{aligned}$$



$$\angle(p_1 - z_2) = \angle(-.457 + 2.096\imath) = 102.3^\circ$$

Obviously,

$$\angle(p_1 - p_2) = 90^\circ$$

$$\angle(p_1 - p_3) = 147.2^\circ, \qquad \angle(p_1 - p_4) = 116.03^\circ$$

Departure Angle

Numerical Examples

Now that we have all the angles:

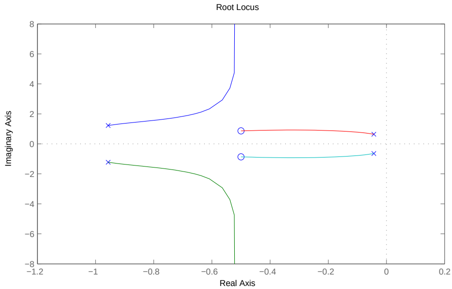
$$\begin{aligned}\angle G(p_1) &= \angle(p_1 - z_1) + \angle(p_1 - z_2) - \angle(p_1 - p_2) - \angle(p_1 - p_3) - \angle(p_1 - p_4)\\ &= 141.46^\circ + 102.3^\circ - 90^\circ - 147.2^\circ - 116.03^\circ\\ &= -109.47^\circ\end{aligned}$$

We conclude

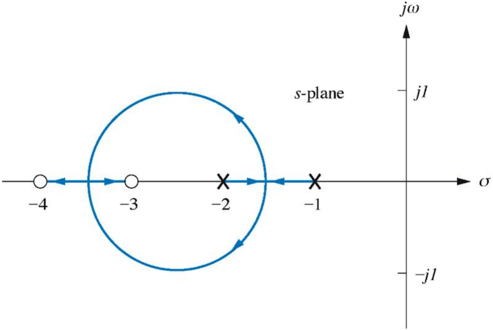
$$\angle_{dep,p_1} = \angle G(p_1) + 180^\circ = 70.53^\circ$$

By symmetry we could find

$$\angle_{dep,p_2} = -70.53^\circ$$



What about a pole on the real axis?

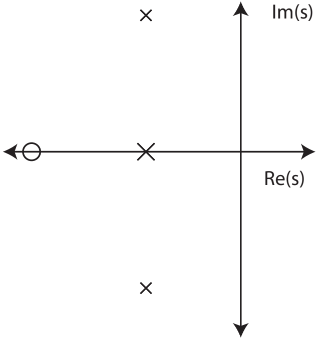


$$\angle G(p) = 0^\circ \quad \text{or} \quad 180^\circ$$

Calculating the Departure Angle

DIY Example

$$G(s) = \frac{s+2}{(s+1)(s^2+2s+2)}$$



Summary

What have we learned today?

Review: What happens at high gain?

- Angles of Departure

The Case of 90° Departure

- Calculating the center of asymptotes

Breaking off the Real Axis

- Break Points

What is the effect of small gain?

- Departure Angles

Next Lecture: Arrival Angles, Summary + Examples