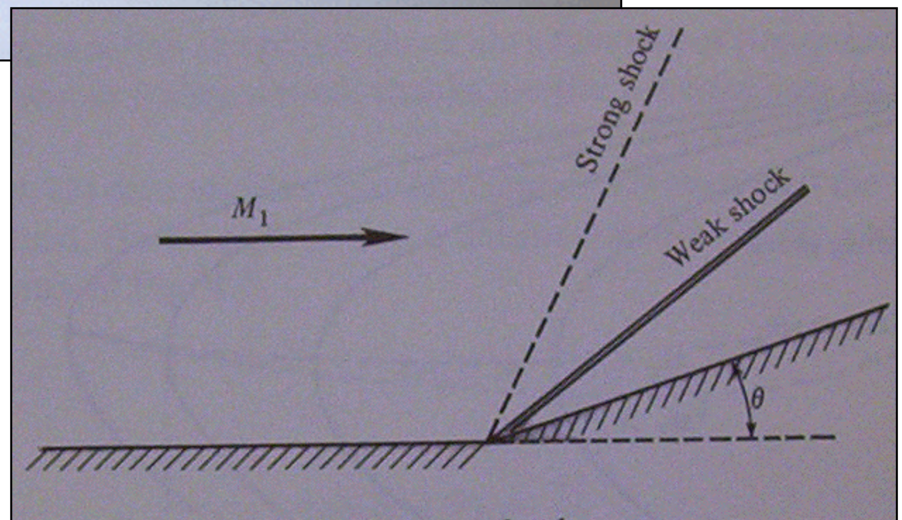
Section 6 Lecture 1:Oblique Shock Waves



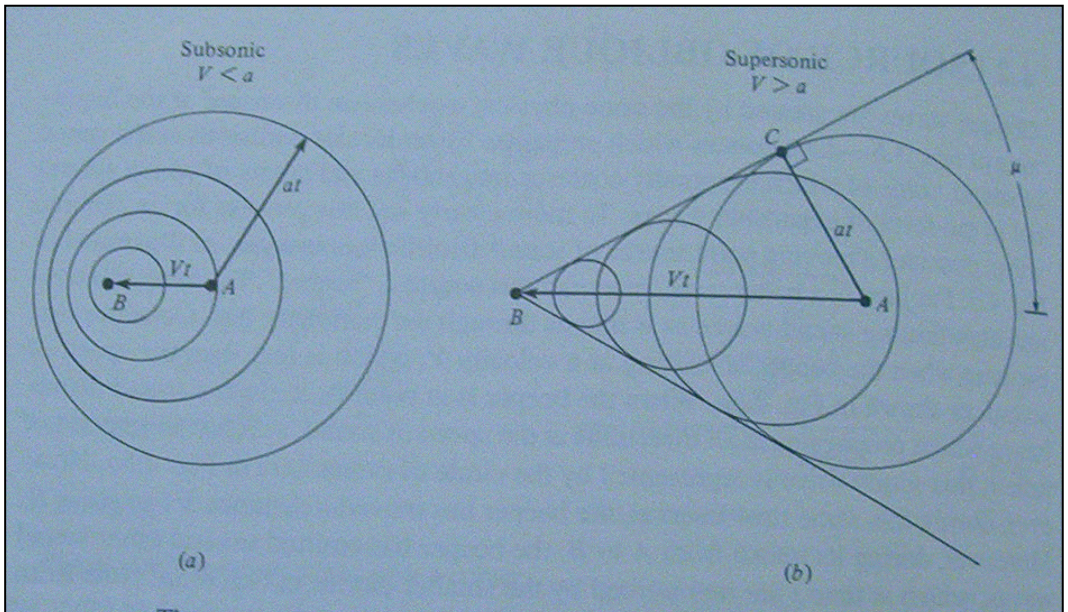
Weak and strong shocks.

- Anderson,
  Chapter 4 pp.127-145

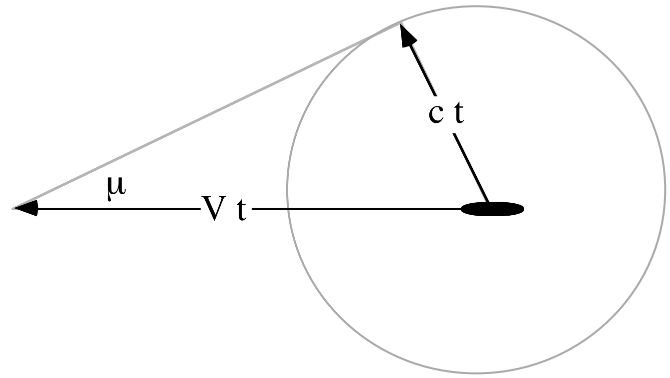
Mach Waves, Revisited

- In Supersonic flow, pressure disturbances cannot outrun "point-mass" generating object

- Result is an infinitesimally weak "mach wave"



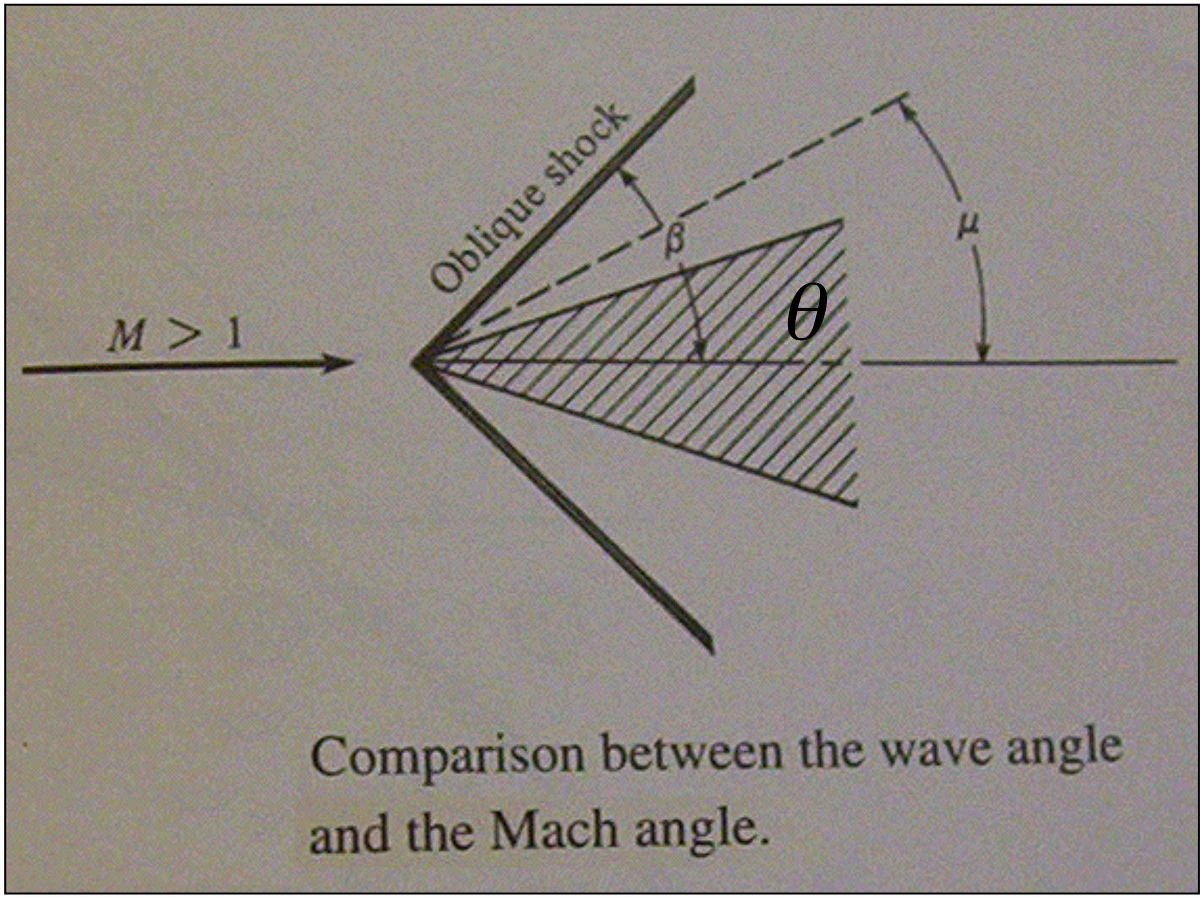
The propagation of disturbances in (a) subsonic and (b) supersonic flow.



$$\sin\mu = \left[\frac{c \times t}{V \times t}\right] = \frac{1}{M} \rightarrow \mu = \sin^{-1}\frac{1}{M}$$

Oblique Shock Wave

- When generating object is larger than a "point", shockwave is stronger than mach wave …. *Oblique shock wave*

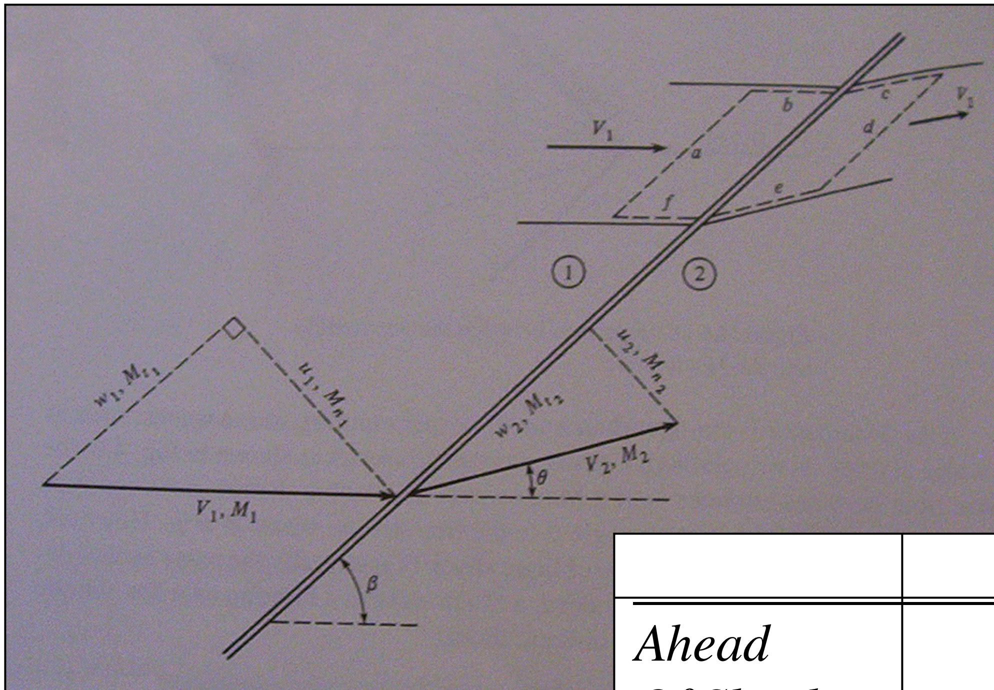


Comparison between the wave angle and the Mach angle.

$$\beta \geq \mu$$

- β -- shock angle
- θ -- turning or "wedge angle"

Oblique Shock Wave Geometry

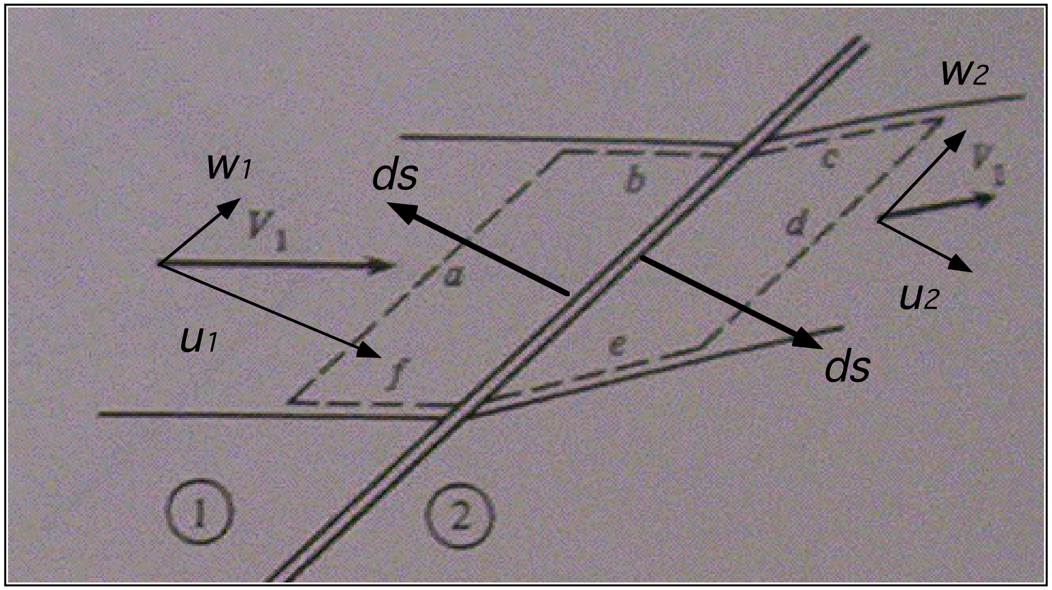


Oblique shock wave geometry.

- Must satisfy
  - i) continuity
  - ii) momentum
  - iii) energy

	Tangential	Normal
Ahead Of Shock	w_1, M_{t_1}	u_1, M_{n_1}
Behind Shock	w_2, M_{t_2}	u_2, M_{n_2}

Continuity Equation



- For Steady Flow

$$-\iint_{C.S.}\left(\rho \vec{V}\bullet \vec{ds}\right) = \frac{\partial}{\partial t}\left(\iiint_{c.v.}\rho dv\right) \rightarrow 0$$

$$\iint_{C.S.}\left(\rho \vec{V}\bullet \vec{ds}\right) = 0 = -\rho_1 u_1 A + \rho_2 u_2 A \rightarrow \boxed{\rho_1 u_1 = \rho_2 u_2}$$

Momentum Equation

- For Steady Flow *w/no* Body Forces

$$\iint_{C.S.} \left(\rho \vec{V} \bullet \vec{ds} \right) \vec{V} = -\iint_{C.S.} (p) \vec{dS}$$

- Tangential Component

$$\left(-\rho_1 u_1 w_1 A + \rho_2 u_2 w_2 A \right) = 0$$

- But from continuity

$$\rho_1 u_1 = \rho_2 u_2 \longrightarrow w_1 = w_2$$

Tangential velocity is Constant across oblique Shock wave

Momentum Equation (concluded)

$$\iint_{C.S.}\left(\rho\vec{V}\bullet\vec{ds}\right)\vec{V} = -\iint_{C.S.}(p)\vec{dS}$$

- Normal Component

Tangential velocity is Constant across oblique Shock wave

$$-\rho_1 u_1^{\,2} A + \rho_2 u_2^{\,2} A = \left(p_2 - p_1\right) A \rightarrow$$

$$\boxed{p_1 + \rho_1 u_1^{\,2} = p_2 + \rho_2 u_2^{\,2}}$$

Energy Equation

- Steady Adiabatic Flow

$$\iint_{C.S.} \rho\left(e+\frac{\left\|V^{2}\right\|}{2}\right)\vec{V}\bullet d\vec{S}+\iint_{C.S.}(pd\vec{S})\bullet\vec{V}=0$$

- Tangential velocity components do not contribute to integrals … thus …

$$p_1u_1+\rho_1\left(e_1+\frac{\left\|V_1^{2}\right\|}{2}\right)u_1=p_2u_2+\rho_2\left(e_2+\frac{\left\|V_2^{2}\right\|}{2}\right)u_2$$

Energy Equation (cont'd)

- Factor out $\{\rho_1, u_1\}$, $\{\rho_2, u_2\}$

$$\left[\left(\frac{p_1}{\rho_1} + e_1\right) + \frac{\|V_1^2\|}{2}\right]\rho_1 u_1 = \left[\left(\frac{p_2}{\rho_2} + e_2\right) + \frac{\|V_2^2\|}{2}\right]\rho_2 u_2$$

- But … $\dfrac{p_1}{\rho_1} + e_1 = R_g T_1 + c_v T_1 = \left(c_p - c_v\right)T_1 + c_v T_1 = c_p T_1 = h_1$

$\dfrac{p_2}{\rho_2} + e_2 = h_2$ … and … $\rho_1 u_1 = \rho_2 u_2$

- … thus … $$h_1 + \frac{\|V_1^2\|}{2} = h_2 + \frac{\|V_2^2\|}{2}$$

Energy Equation (concluded)

- Write Velocity in terms of components

$$\left\|V_1^{\,2}\right\| = u_1^{\,2} + w_1^{\,2} \rightarrow \left\|V_2^{\,2}\right\| = u_2^{\,2} + w_2^{\,2} \rightarrow w_1 = w_2$$

- thus …

$$\boxed{h_1 + \frac{u_1^{\,2}}{2} = h_2 + \frac{u_2^{\,2}}{2}}$$

Collected Oblique Shock Equations

- Continuity

$$\rho_1 u_1 = \rho_2 u_2$$

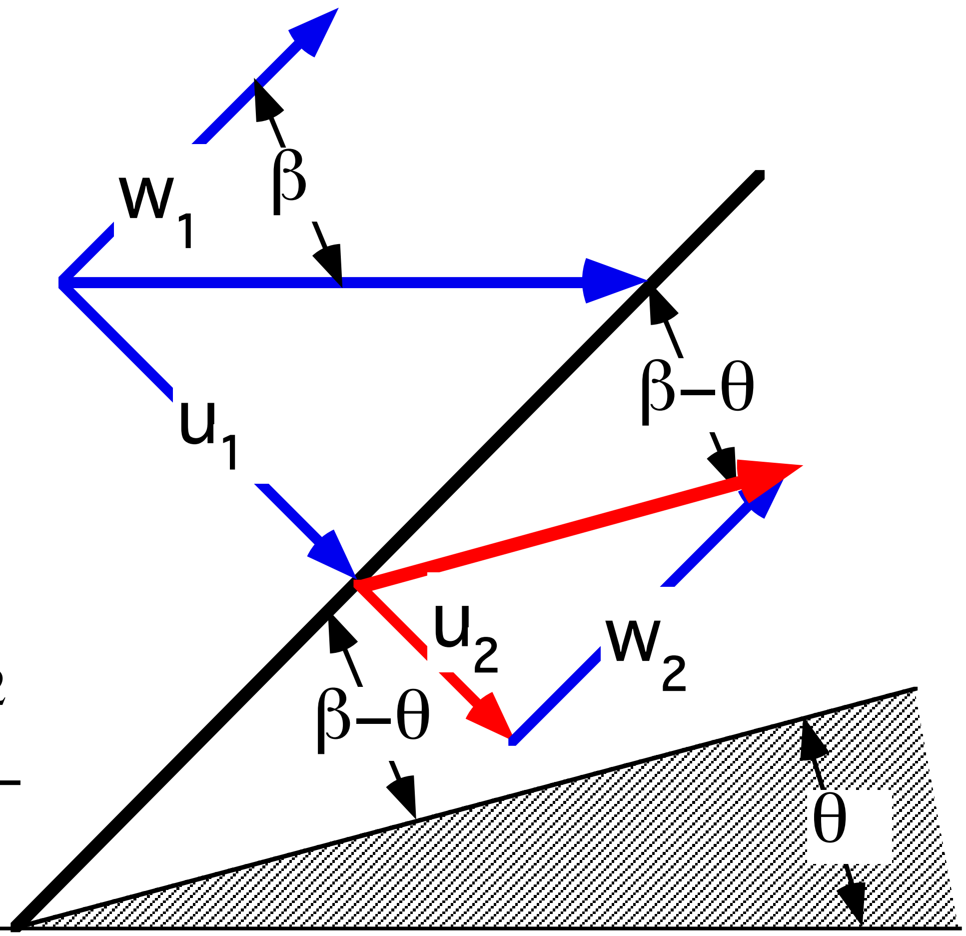
- Momentum

$$w_1 = w_2$$

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$$

- Energy

$$c_p T_1 + \frac{u_1^2}{2} = c_p T_2 + \frac{u_2^2}{2}$$



Compare Oblique to Normal Shock Equations

Normal Shock Equations

- Continuity

$$\rho_1 V_1 = \rho_2 V_2$$

- Momentum

$$p1 + \rho_1 V_1^{\ 2} = p_2 + \rho_2 V_2^{\ 2}$$

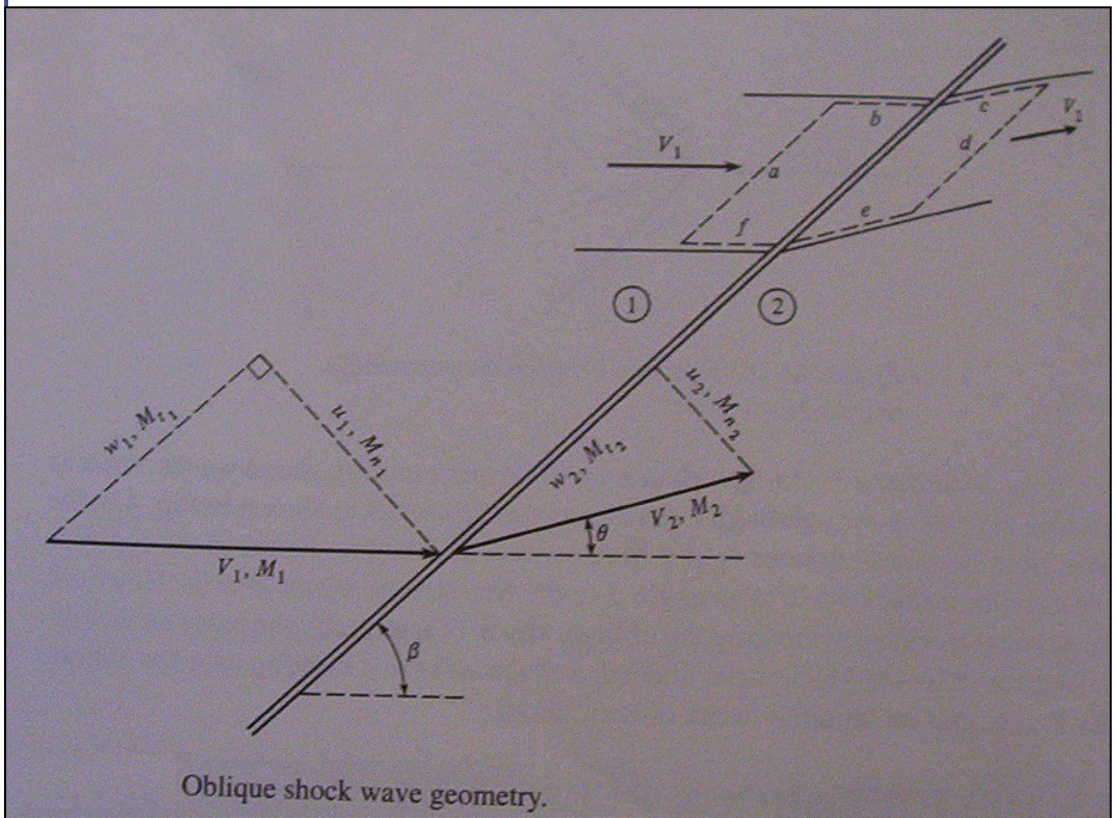
- Energy

$$c_{p1} T_1 + \frac{V_1^{\ 2}}{2} = c_{p2} T_2 + \frac{V_2^{\ 2}}{2}$$

- Identical except for u_1 replaces V_1 *(normal to shock wave)* and $w_1 = w_2$ *(tangential to shock wave)*

Compare Oblique to Normal Shock Equations

(cont'd)



Oblique shock wave geometry.

- Defining:

$$M_{n_1} = M_1 \sin(\beta)$$

$$M_{t_1} = M_1 \cos(\beta)$$

- Then by similarity we can write the solution

$$M_{n_2} = \sqrt{\frac{\left(1 + \frac{(\gamma - 1)}{2} M_{n_1}^2\right)}{\left(\gamma M_{n_1}^2 - \frac{(\gamma - 1)}{2}\right)}}$$

Compare Oblique to Normal Shock Equations

(cont'd)

- Similarity Solution

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1) Mn_1^{\ 2}}{\left(2 + (\gamma - 1) Mn_1^{\ 2}\right)}$$

Letting

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{(\gamma + 1)}\left(Mn_1^{\ 2} - 1\right) \longrightarrow Mn_1 = M_1 \sin(\beta)$$

$$\frac{T_2}{T_1} = \left[1 + \frac{2\gamma}{(\gamma + 1)}\left(Mn_1^{\ 2} - 1\right)\right]\left[\frac{\left(2 + (\gamma - 1) Mn_1^{\ 2}\right)}{(\gamma + 1) Mn_1^{\ 2}}\right]$$

Then …..

Compare Oblique to Normal Shock Equations

(cont'd)

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)(M_1 \sin\beta)^2}{\left(2+(\gamma-1)(M_1 \sin\beta)^2\right)}$$

$$M_{n_2} = \sqrt{\frac{\left(1+\frac{(\gamma-1)}{2}(M_1 \sin\beta)^2\right)}{\left(\gamma (M_1 \sin\beta)^2 - \frac{(\gamma-1)}{2}\right)}}$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{(\gamma+1)}\left((M_1 \sin\beta)^2 - 1\right)$$

$$\frac{T_2}{T_1} = \left[1 + \frac{2\gamma}{(\gamma+1)}\left((M_1 \sin\beta)^2 - 1\right)\right]\left[\frac{\left(2+(\gamma-1)(M_1 \sin\beta)^2\right)}{(\gamma+1)(M_1 \sin\beta)^2}\right]$$

- **Properties across Oblique Shock wave ~*f*(M_1, β)**

Total Mach Number Downstream of Oblique Shock

Tangential velocity is Constant across oblique Shock wave

$$\boxed{w_1 = w_2}$$

$$w_1 = w_2 \rightarrow Mt_1 c_1 = Mt_2 c_2 = M_1 \cos(\beta) c_1$$

$$Mt_2 = \frac{M_1 \cos(\beta) c_1}{c_2} = M_1 \cos(\beta) \sqrt{\frac{T_1}{T_2}}$$

$$M_2 = \sqrt{\left[Mt_2{}^2 + Mn_2{}^2 \right]}$$

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Total Mach Number Downstream of Oblique Shock (cont'd)

Tangential velocity is Constant across oblique Shock wave

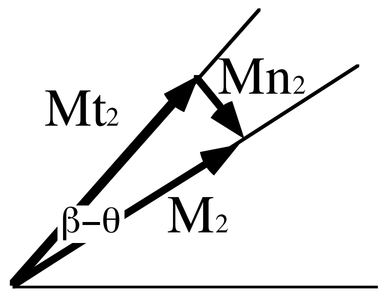
$$M_2 = \sqrt{\left[Mt_2^{\ 2} + Mn_2^{\ 2}\right]} \rightarrow Mn_2 = \sqrt{\frac{\left(1 + \frac{(\gamma - 1)}{2}\left[M_1 \sin(\beta)\right]^2\right)}{\left(\gamma\left[M_1 \sin(\beta)\right]^2 - \frac{(\gamma - 1)}{2}\right)}}$$

$$M_2 = \sqrt{\left[\left[M_1 \cos(\beta)\right]^2 \frac{T_1}{T_2} + \frac{\left(1 + \frac{(\gamma - 1)}{2}\left[M_1 \sin(\beta)\right]^2\right)}{\left(\gamma\left[M_1 \sin(\beta)\right]^2 - \frac{(\gamma - 1)}{2}\right)}\right]}$$

Total Mach Number Downstream of Oblique Shock (concluded)

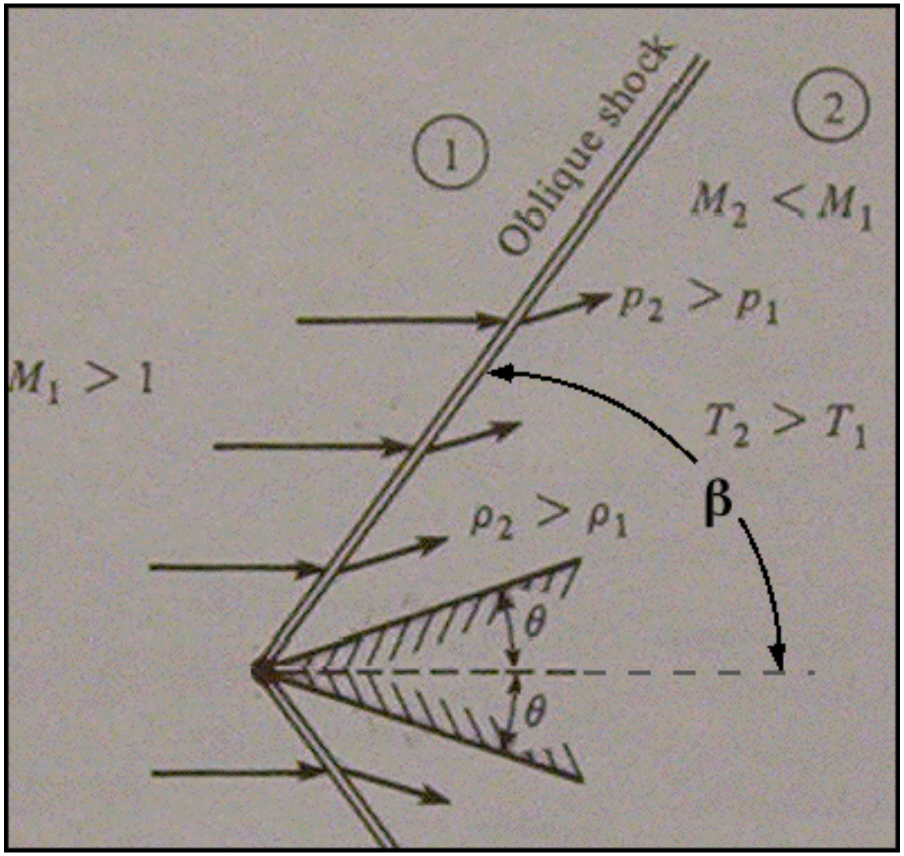
Tangential velocity is Constant across oblique Shock wave

- Or … More simply .. If we consider geometric arguments



$$M_2 = \frac{Mn_2}{\sin(\beta - \theta)}$$

Oblique Shock Wave Angle



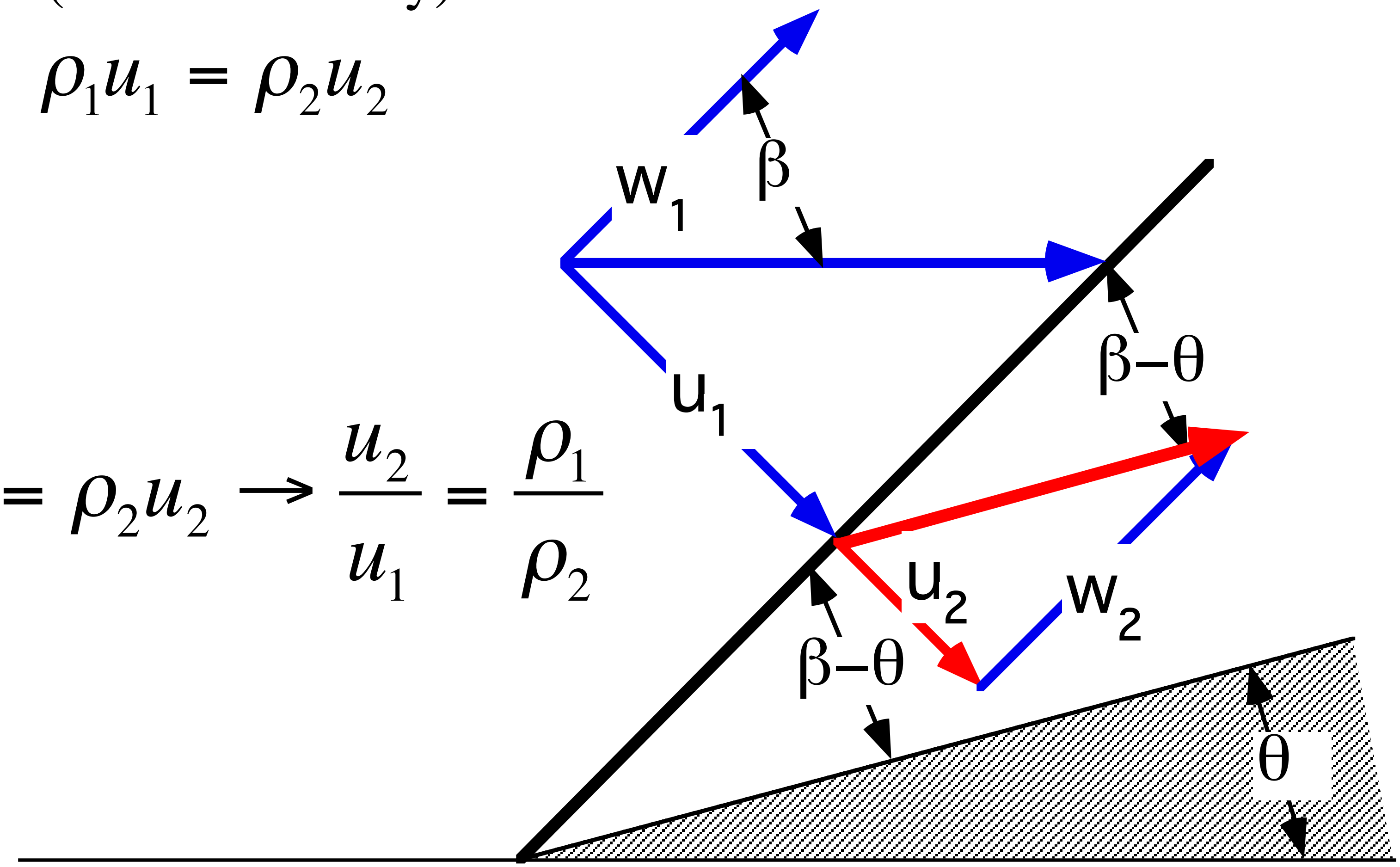
- **Properties across Oblique Shock wave ~*f*(M_1, β)**
- **θ is the geometric angle that "forces" the flow**
- **How do we relate θ *to* β ?**

Oblique Shock Wave Angle (cont'd)

- Since (from continuity)

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1 = \rho_2 u_2 \rightarrow \frac{u_2}{u_1} = \frac{\rho_1}{\rho_2}$$

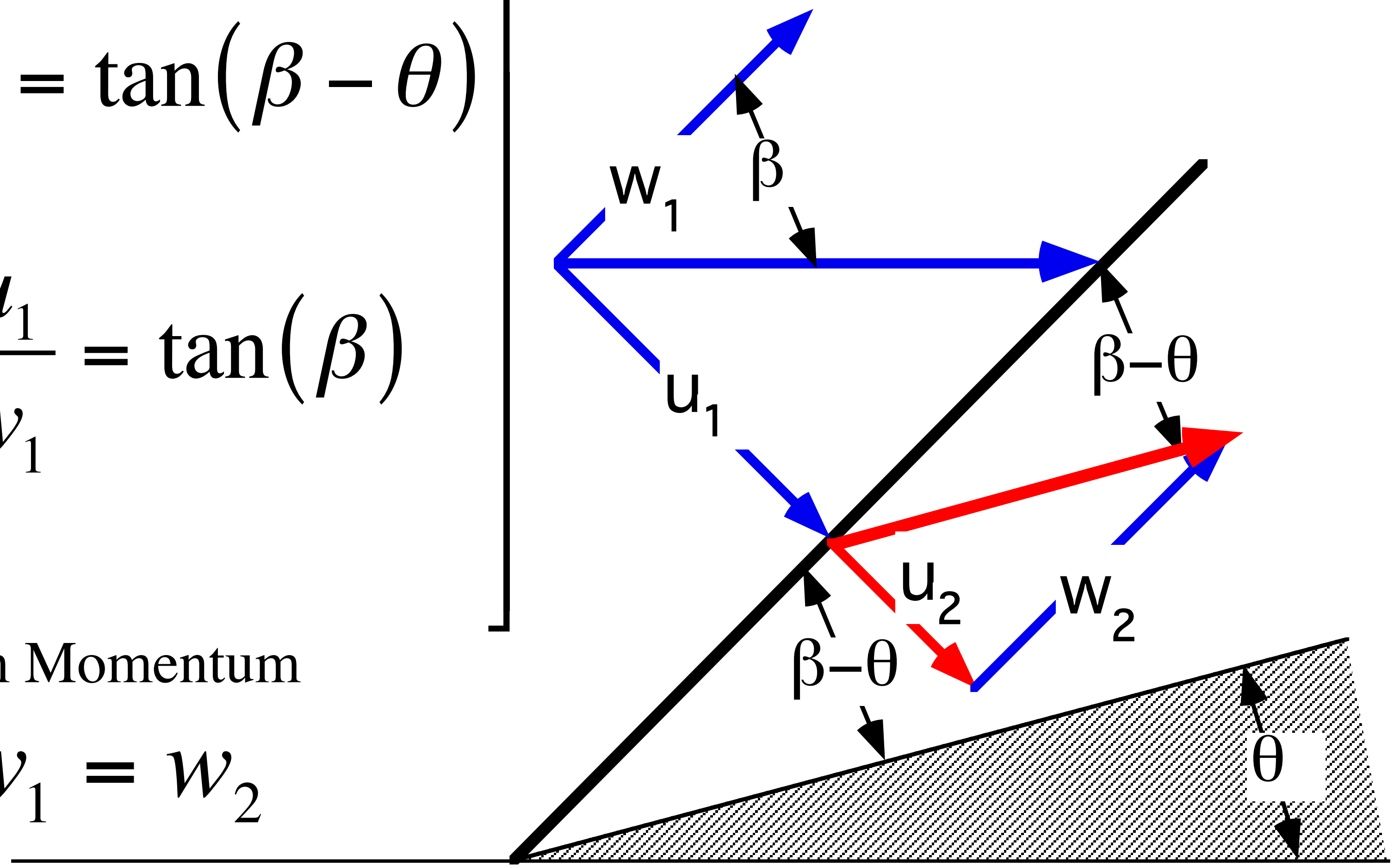


Oblique Shock Wave Angle (cont'd)

$$\left[\begin{array}{l} \dfrac{u_2}{w_2} = \tan(\beta - \theta) \\ \dfrac{u_1}{w_1} = \tan(\beta) \end{array}\right]$$

- from Momentum

$$w_1 = w_2$$



Oblique Shock Wave Angle (cont'd)

- Solving for the ratio u_2/u_1

$$\rightarrow \frac{u_2}{u_1} = \frac{\tan(\beta - \theta)}{\tan(\beta)} = \frac{\rho_1}{\rho_2} \rightarrow \rightarrow \frac{\rho_2}{\rho_1} = \frac{(\gamma + 1) Mn_1^{\;2}}{\left(2 + (\gamma - 1) Mn_1^{\;2}\right)}$$

$$\therefore \frac{\tan(\beta - \theta)}{\tan(\beta)} = \frac{\left(2 + (\gamma - 1)\left[M_1 \sin(\beta)\right]^2\right)}{(\gamma + 1)\left[M_1 \sin(\beta)\right]^2}$$

Implicit relationship for shock angle in terms of Free stream mach number and "wedge angle"

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Oblique Shock Wave Angle (cont'd)

- Solve explicitly for tan(θ)

$$\frac{\tan(\beta-\theta)}{\tan(\beta)} = \frac{\sin(\beta-\theta)}{\cos(\beta-\theta)} = \left(\frac{\sin(\beta)\cos(\theta)-\cos(\beta)\sin(\theta)}{\cos(\beta)\cos(\theta)+\sin(\beta)\sin(\theta)}\right)\frac{\cos\beta}{\sin\beta} =$$

$$\left(\frac{\dfrac{\sin(\beta)\cos(\theta)}{\sin\beta}-\dfrac{\cos(\beta)\sin(\theta)}{\sin\beta}}{\dfrac{\cos(\beta)\cos(\theta)}{\cos\beta}+\dfrac{\sin(\beta)\sin(\theta)}{\cos\beta}}\right) = \left(\frac{\cos(\theta)-\dfrac{\sin(\theta)}{\tan(\beta)}}{\cos(\theta)+\tan(\beta)\sin(\theta)}\right) =$$

$$\left(\frac{1-\dfrac{\sin(\theta)}{\cos(\theta)\tan(\beta)}}{1+\dfrac{\tan(\beta)\sin(\theta)}{\cos(\theta)}}\right) = \left(\frac{1-\dfrac{\tan(\theta)}{\tan(\beta)}}{1+\tan(\beta)\tan(\theta)}\right) = \frac{\tan(\beta)-\tan(\theta)}{\tan(\beta)+\tan^2(\beta)\tan(\theta)}$$

Oblique Shock Wave Angle (cont'd)

- Solve explicitly for tan(θ)

$$\frac{\tan(\beta)-\tan(\theta)}{\tan(\beta)+\tan^2(\beta)\tan(\theta)}=\frac{\left(2+(\gamma-1)\left[M_1\sin(\beta)\right]^2\right)}{(\gamma+1)\left[M_1\sin(\beta)\right]^2}$$

$\longrightarrow$

Oblique Shock Wave Angle (cont'd)

- Solve for tan(θ)

$$\left[\tan(\beta)-\tan(\theta)\right](\gamma+1)\left[M_1\sin(\beta)\right]^2 =$$

$$\left[\tan(\beta)+\tan^2(\beta)\tan(\theta)\right]\left(2+(\gamma-1)\left[M_1\sin(\beta)\right]^2\right)\rightarrow$$

$$\tan(\beta)\left[(\gamma+1)\left[M_1\sin(\beta)\right]^2-\left(2+(\gamma-1)\left[M_1\sin(\beta)\right]^2\right)\right]=$$

$$\tan(\theta)\left[(\gamma+1)\left[M_1\sin(\beta)\right]^2+\tan^2(\beta)\left(2+(\gamma-1)\left[M_1\sin(\beta)\right]^2\right)\right]\rightarrow$$

$$\tan(\theta)=\frac{\tan(\beta)\left[(\gamma+1)\left[M_1\sin(\beta)\right]^2-\left(2+(\gamma-1)\left[M_1\sin(\beta)\right]^2\right)\right]}{\left[(\gamma+1)\left[M_1\sin(\beta)\right]^2+\tan^2(\beta)\left(2+(\gamma-1)\left[M_1\sin(\beta)\right]^2\right)\right]}$$

Oblique Shock Wave Angle (cont'd)

- Simplify Numerator

$$\tan(\beta)\left[(\gamma+1)\left[M_1\sin(\beta)\right]^2-\left(2+(\gamma-1)\left[M_1\sin(\beta)\right]^2\right)\right]=$$

$$\tan(\beta)\left[\gamma\left[M_1\sin(\beta)\right]^2+\left[M_1\sin(\beta)\right]^2-2-\gamma\left[M_1\sin(\beta)\right]^2+\left[M_1\sin(\beta)\right]^2\right]=$$

$$\boxed{\tan(\beta)\left[2\left\{\left[M_1\sin(\beta)\right]^2-1\right\}\right]}$$

Oblique Shock Wave Angle (cont'd)

- Simplify Denominator

$$\left[(\gamma+1)\left[M_1\sin(\beta)\right]^2+\tan^2(\beta)\left(2+(\gamma-1)\left[M_1\sin(\beta)\right]^2\right)\right]=$$

$$\tan^2(\beta)\left[(\gamma+1)\left[M_1\frac{\sin(\beta)}{\tan(\beta)}\right]^2+\left(2+(\gamma-1)\left[M_1\sin(\beta)\right]^2\right)\right]=$$

$$\tan^2(\beta)\left[(\gamma+1)\left[M_1\cos(\beta)\right]^2+\left(2+(\gamma-1)\left[M_1\sin(\beta)\right]^2\right)\right]=$$

$$\tan^2(\beta)\left[(\gamma+1)M_1^{\,2}\left[1-\sin^2(\beta)\right]+2+(\gamma-1)M_1^{\,2}\sin^2(\beta)\right]=$$

$$\tan^2(\beta)\left[2+(\gamma+1)M_1^{\,2}-(\gamma+1)M_1^{\,2}\sin^2(\beta)+(\gamma-1)M_1^{\,2}\left[\sin^2(\beta)\right]\right]=$$

$$\tan^2(\beta)\left[2+(\gamma+1)M_1^{\,2}-2M_1^{\,2}\sin^2(\beta)\right]=\tan^2(\beta)\left[2+(\gamma+1)M_1^{\,2}-2M_1^{\,2}\sin^2(\beta)\right]=$$

$$\tan^2(\beta)\left[2+\gamma M_1^{\,2}+M_1^{\,2}\left[1-2\sin^2(\beta)\right]\right]=\tan^2(\beta)\left[2+\gamma M_1^{\,2}+M_1^{\,2}\left[\cos^2(\beta)-\sin^2(\beta)\right]\right]=$$

$$\boxed{\tan^2(\beta)\left[2+M_1^{\,2}\left[\gamma+\cos(2\beta)\right]\right]}$$

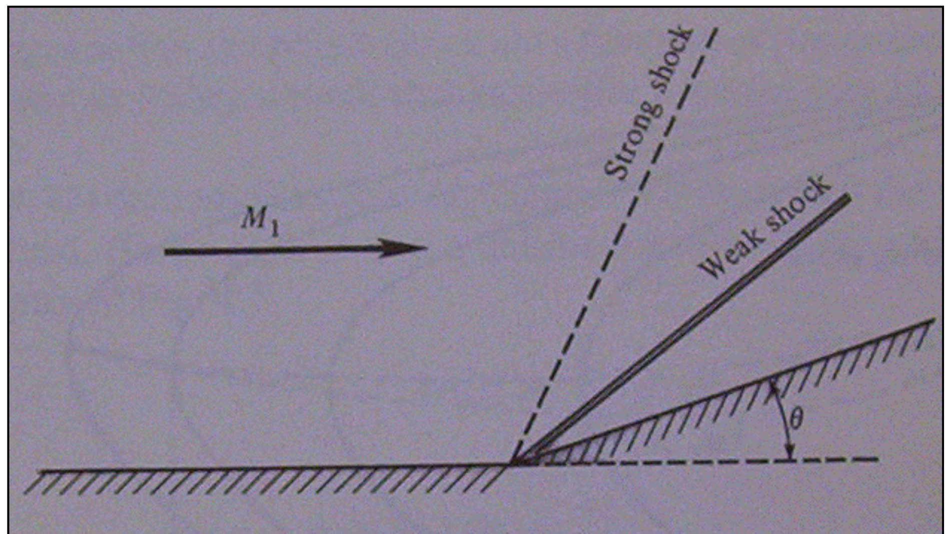
Oblique Shock Wave Angle (cont'd)

- Collect terms

$$\tan(\theta) = \frac{2\tan(\beta)\left\{\left[M_1 \sin(\beta)\right]^2 - 1\right\}}{\tan^2(\beta)\left[2 + M_1^{\,2}\left[\gamma + \cos(2\beta)\right]\right]} = \frac{2\left\{M_1^{\,2}\sin^2(\beta) - 1\right\}}{\tan(\beta)\left[2 + M_1^{\,2}\left[\gamma + \cos(2\beta)\right]\right]}$$

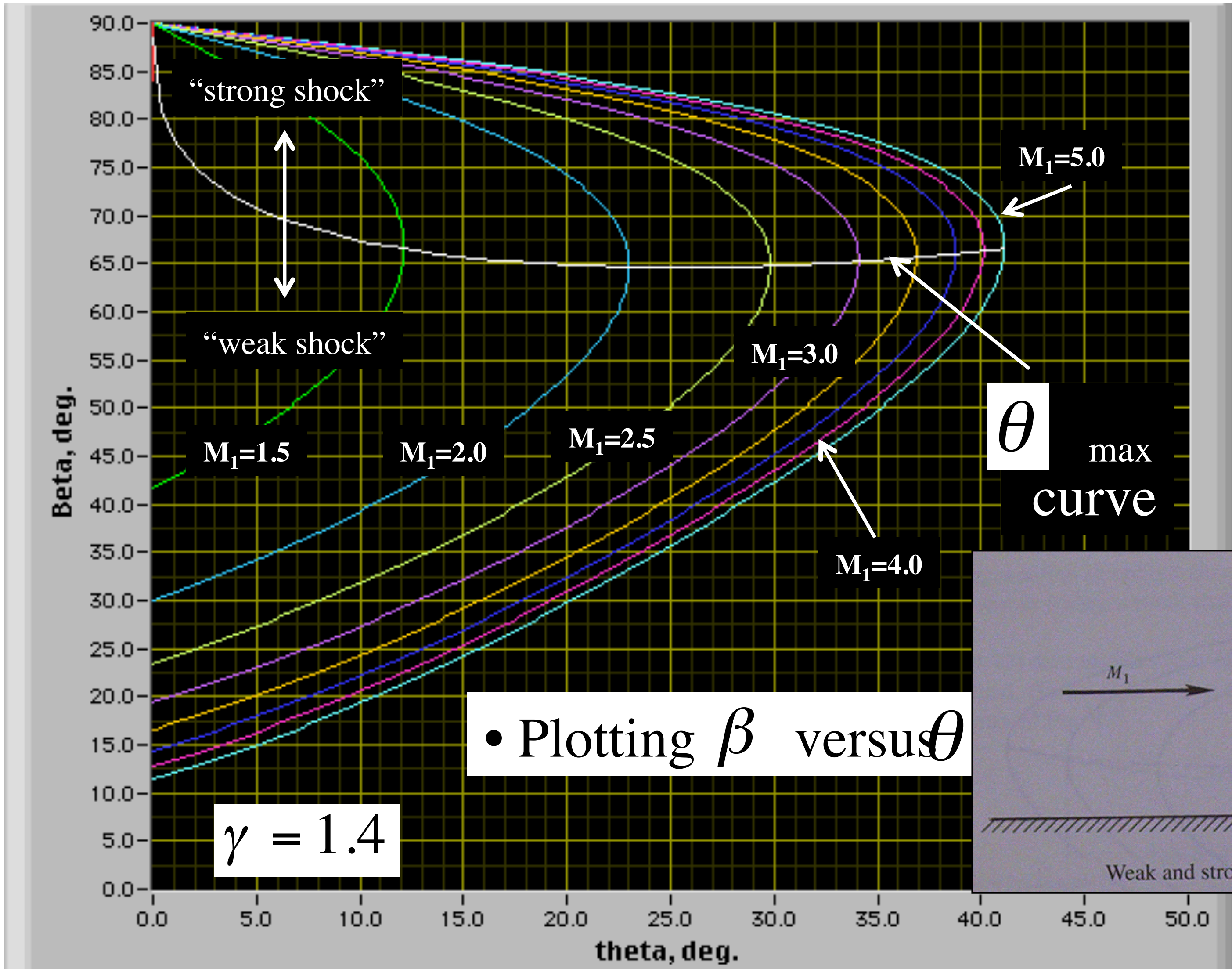
- "Wedge Angle" Given explicitly as function of shock angle and freestream Mach number

- Two Solutions "weak" and "strong" shock wave … in reality weak shock typically occurs; strong only occurs under very Specialized circumstances .e.g near stagnation point for a detached Shock (Anderson, pp. 138-139, 165,166)

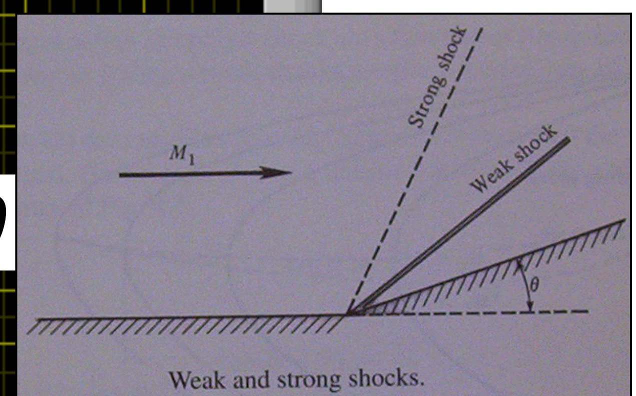


Weak and strong shocks.

Oblique Shock Wave Angle (concluded)



- Plotting β versus θ



Weak and strong shocks.

Compare Oblique to Normal Shock Equations

(cont'd)

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)(M_1 \sin\beta)^2}{\left(2+(\gamma-1)(M_1 \sin\beta)^2\right)}$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{(\gamma+1)}\left((M_1 \sin\beta)^2 - 1\right)$$

$$M_{n_2} = \sqrt{\frac{\left(1+\frac{(\gamma-1)}{2}(M_1 \sin\beta)^2\right)}{\left(\gamma(M_1 \sin\beta)^2 - \frac{(\gamma-1)}{2}\right)}}$$

$$\frac{T_2}{T_1} = \left[1 + \frac{2\gamma}{(\gamma+1)}\left((M_1 \sin\beta)^2 - 1\right)\right]\left[\frac{\left(2+(\gamma-1)(M_1 \sin\beta)^2\right)}{(\gamma+1)(M_1 \sin\beta)^2}\right]$$

- **Properties across Oblique Shock wave ~** $f(M_1, \beta)$

Solving for Oblique Shock Wave Angle in Terms of Wedge Angle

- As derived

$$\tan(\theta) = \frac{2\left\{M_1^{\,2}\sin^2(\beta) - 1\right\}}{\tan(\beta)\left[2 + M_1^{\,2}\left[\gamma + \cos(2\beta)\right]\right]}$$

- "Wedge Angle" Given explicitly as function of shock angle and freestream Mach number

- For most practical applications, the geometric deflection angle *(wedge angle)* and Mach number are prescribed .. Need β in terms of θ and M_1

- Obvious Approach …. Numerical Solution using Newton's method

Solving for Oblique Shock Wave Angle in Terms of Wedge Angle (cont'd)

- Newton method

$$\frac{2\left\{M_1^{\,2}\sin^2\left(\beta\right)-1\right\}}{\tan\left(\beta\right)\left[2+M_1^{\,2}\left[\gamma+\cos\left(2\beta\right)\right]\right]}-\tan\left(\theta\right)\equiv f(\beta)=0$$

$$f(\beta)=f(\beta_{(j)})+\left(\frac{\partial f}{\partial \beta}\right)_{(j)}\left(\beta-\beta_{(j)}\right)+O(\beta^2)+\ldots.\rightarrow$$

$$\beta_{(j+1)}=\beta_{(j)}-\frac{\dfrac{2\left\{M_1^{\,2}\sin^2\left(\beta\right)-1\right\}}{\tan\left(\beta\right)\left[2+M_1^{\,2}\left[\gamma+\cos\left(2\beta\right)\right]\right]}-\tan\left(\theta\right)}{\left(\dfrac{\partial f}{\partial \beta}\right)_{(j)}}$$

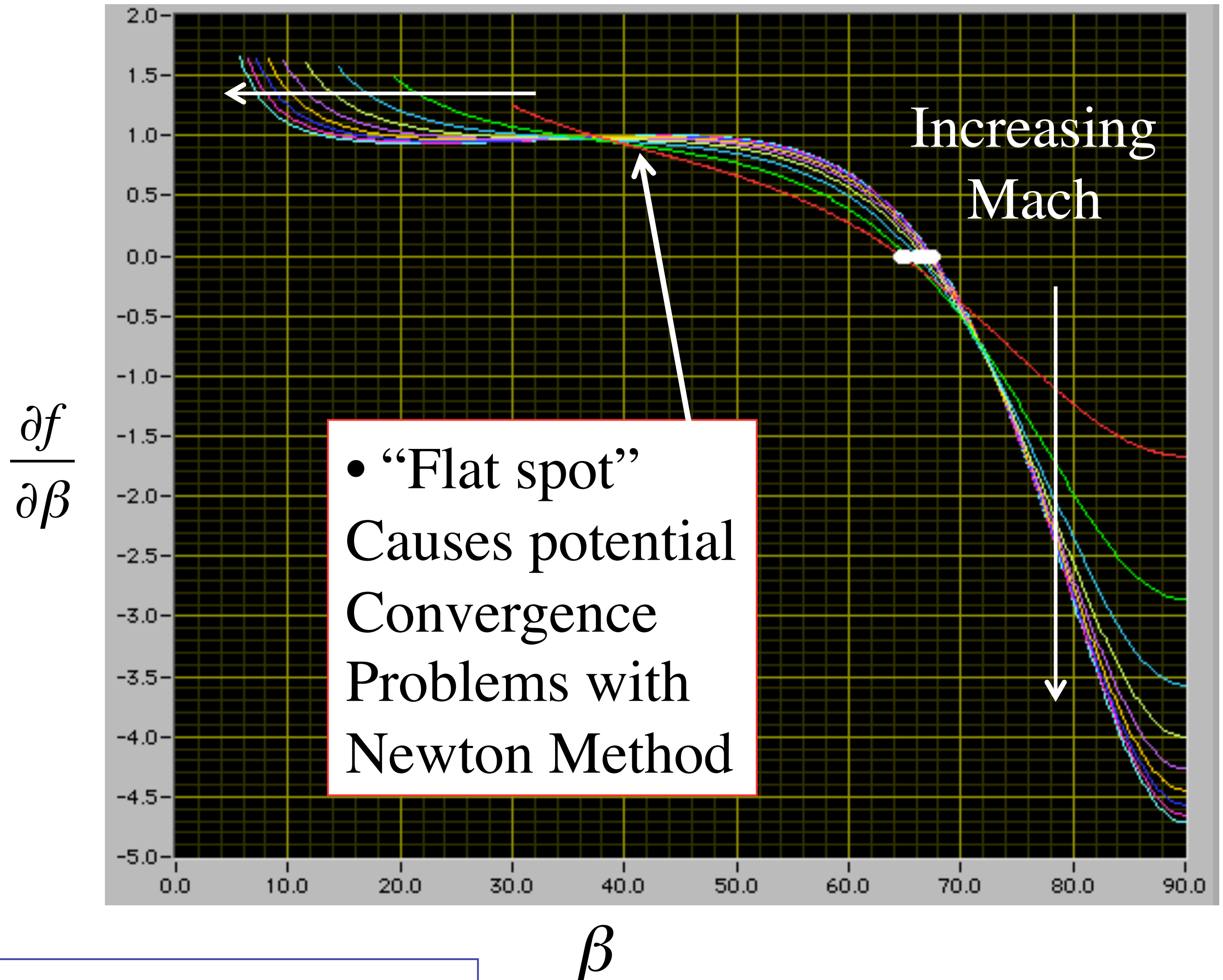
Solving for Oblique Shock Wave Angle in Terms of Wedge Angle (cont'd)

- Newton method (continued)

$$\frac{\partial f}{\partial \beta} = \frac{2\left[M_1^4 \sin^2(\beta)\left[1+\gamma\cos(2\beta)\right] + M_1^2\left(2\cos(2\beta)+\gamma-1\right)+2\right]}{\sin^2(\beta)\left[2+M_1^2\left[\gamma+\cos(2\beta)\right]\right]^2}$$

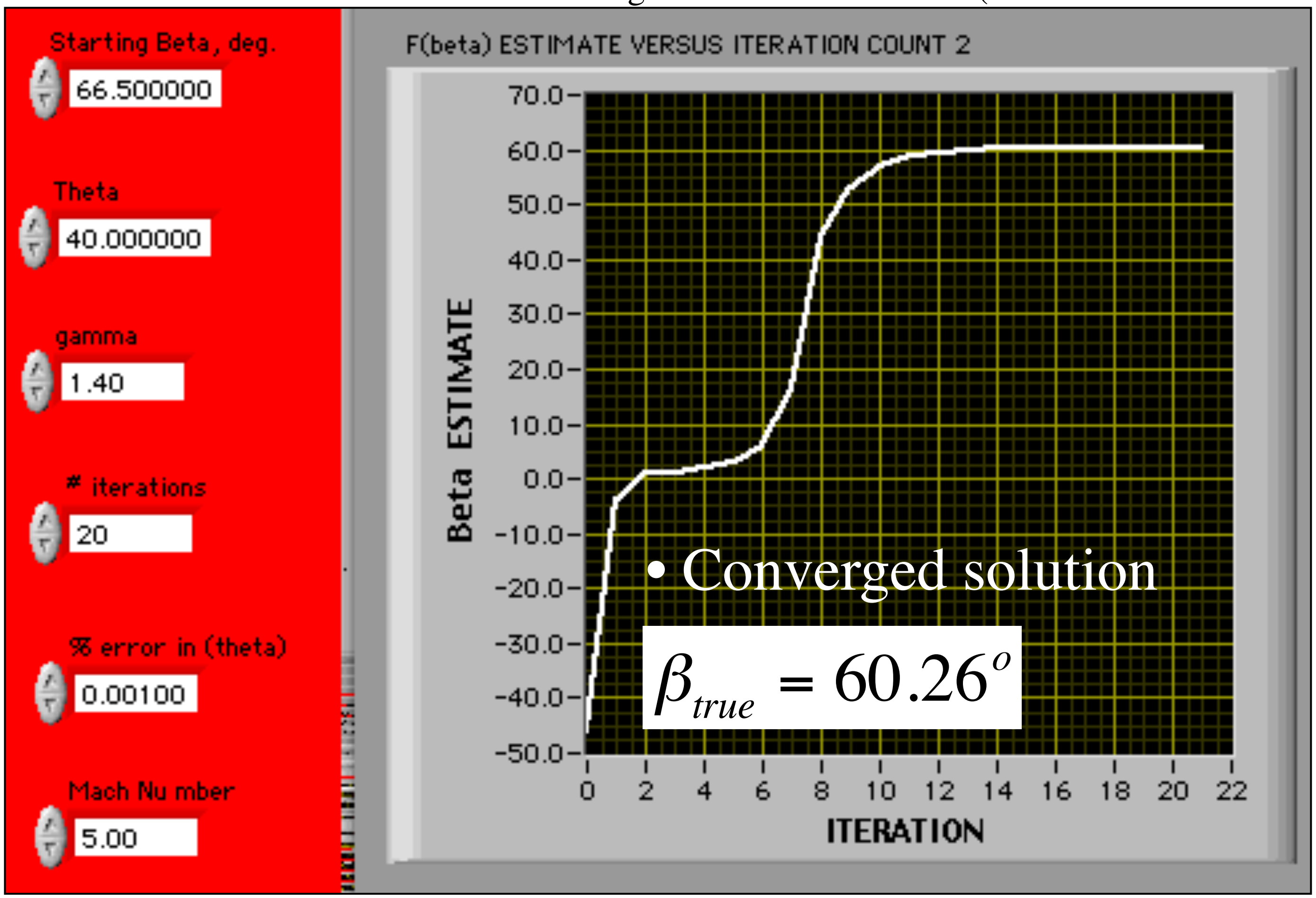
- Iterate until convergence

Solving for Oblique Shock Wave Angle in Terms of Wedge Angle (cont'd)



Solving for Oblique Shock Wave Angle in Terms of Wedge Angle (cont'd)

• Newton method … Convergence can often be slow (because of low derivative slope)

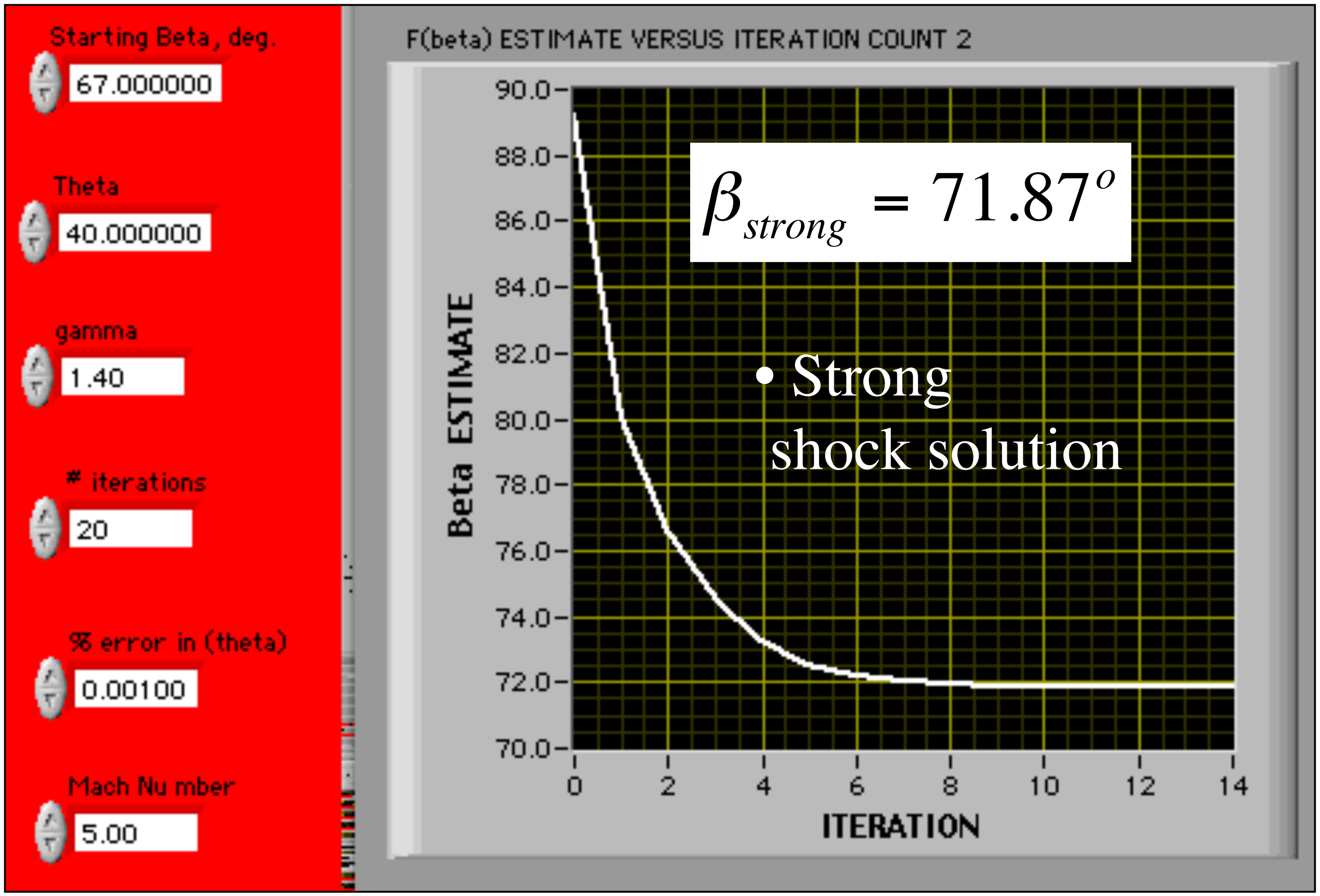


• Converged solution

$$\beta_{true} = 60.26^o$$

Solving for Oblique Shock Wave Angle in Terms of Wedge Angle (concluded)

- Newton method … or can "toggle" to strong shock solution



Solving for Oblique Shock Wave Angle in Terms of Wedge Angle (improved solution)

- Because of the slow convergence of Newton's method for this implicit function… explicit solution … (if possible) .. Or better behaved .. Method *very desirable*

Substitute $\cos(2\beta) = \cos^2(\beta) - \sin^2(\beta)$

$$\tan(\theta) = \frac{2\left\{M_1^{\,2}\sin^2(\beta) - 1\right\}}{\tan(\beta)\left[2 + M_1^{\,2}\left[\gamma + \cos(2\beta)\right]\right]} =$$

$$\frac{2\left\{M_1^{\,2}\sin^2(\beta) - 1\right\}}{\tan(\beta)\left[2 + \gamma M_1^{\,2} + M_1^{\,2}\left[\cos^2(\beta) - \sin^2(\beta)\right]\right]}$$

Solving for Oblique Shock Wave Angle in Terms of Wedge Angle (improved solution)

(cont'd)

- But, since $1 = \cos^2(\beta) + \sin^2(\beta)$

$$\frac{2\left\{M_1{}^2 \sin^2(\beta) - 1\right\}}{\tan(\beta)\left[2 + \gamma M_1{}^2 + M_1{}^2\left[\cos^2(\beta) - \sin^2(\beta)\right]\right]} = \frac{2\left\{M_1{}^2 \sin^2(\beta) - \sin^2(\beta) - \cos^2(\beta)\right\}}{\tan(\beta)\left[2 + \gamma M_1{}^2 + M_1{}^2\left[\cos^2(\beta) - \sin^2(\beta)\right]\right]}$$

Solving for Oblique Shock Wave Angle in Terms of Wedge Angle (improved solution)

(cont'd)

- Simplify and collect terms

$$\frac{2\left\{M_1^{\,2}\sin^2(\beta)-\sin^2(\beta)-\cos^2(\beta)\right\}}{\tan(\beta)\left[2+\gamma M_1^{\,2}+M_1^{\,2}\left[\cos^2(\beta)-\sin^2(\beta)\right]\right]}=$$

$$\frac{\left\{\left(M_1^{\,2}-1\right)\sin^2(\beta)-\cos^2(\beta)\right\}}{\tan(\beta)\left[1+\frac{\gamma}{2}M_1^{\,2}+\frac{1}{2}M_1^{\,2}\left[\cos^2(\beta)-\sin^2(\beta)\right]\right]}=$$

$$\frac{\left\{\left(M_1^{\,2}-1\right)\sin^2(\beta)-\cos^2(\beta)\right\}}{\tan(\beta)\left[1+\frac{\gamma}{2}M_1^{\,2}+\frac{1}{2}M_1^{\,2}\left[\cos^2(\beta)-\sin^2(\beta)\right]\right]}=$$

$$\frac{\left\{\left(M_1^{\,2}-1\right)\sin^2(\beta)-\cos^2(\beta)\right\}}{\tan(\beta)\left[1+\frac{\gamma+\cos^2(\beta)-\sin^2(\beta)}{2}M_1^{\,2}\right]}$$

Solving for Oblique Shock Wave Angle in Terms of Wedge Angle (improved solution)

(cont'd)

- Again, Since $1 = \cos^2(\beta) + \sin^2(\beta)$

$$\frac{\left\{\left(M_1^{\,2}-1\right)\sin^2(\beta)-\cos^2(\beta)\right\}}{\tan(\beta)\left[1+\frac{\gamma+\cos^2(\beta)-\sin^2(\beta)}{2}M_1^{\,2}\right]} =$$

$$\frac{\left\{\left(M_1^{\,2}-1\right)\sin^2(\beta)-\cos^2(\beta)\right\}}{\tan(\beta)\left[\cos^2(\beta)+\sin^2(\beta)+\frac{\gamma\left[\cos^2(\beta)+\sin^2(\beta)\right]+\cos^2(\beta)-\sin^2(\beta)}{2}M_1^{\,2}\right]}$$

Solving for Oblique Shock Wave Angle in Terms of Wedge Angle (improved solution)

(cont'd)

- Regroup and collect terms

$$\frac{\left\{\left(M_1^2-1\right)\sin^2(\beta)-\cos^2(\beta)\right\}}{\tan(\beta)\left[\cos^2(\beta)+\sin^2(\beta)+\frac{\gamma\left[\cos^2(\beta)+\sin^2(\beta)\right]+\cos^2(\beta)-\sin^2(\beta)}{2}M_1^2\right]}=$$

$$\frac{\left\{\left(M_1^2-1\right)\tan^2(\beta)-1\right\}}{\tan(\beta)\left[1+\tan^2(\beta)+\frac{\gamma\left[1+\tan^2(\beta)\right]+1-\tan^2(\beta)}{2}M_1^2\right]}=$$

$$\frac{\left\{\left(M_1^2-1\right)\tan^2(\beta)-1\right\}}{\tan(\beta)\left[1+\frac{\gamma+1}{2}M_1^2+\tan^2(\beta)+\frac{\gamma\left[\tan^2(\beta)\right]-\tan^2(\beta)}{2}M_1^2\right]}=$$

$$\frac{\left\{\left(M_1^2-1\right)\tan^2(\beta)-1\right\}}{\tan(\beta)\left[\left[1+\frac{\gamma+1}{2}M_1^2\right]+\tan^2(\beta)\left[1+\frac{\gamma-1}{2}M_1^2\right]\right]}$$

Solving for Oblique Shock Wave Angle in Terms of Wedge Angle (improved solution)

(cont'd)

- Finally

$$\tan(\theta) = \frac{\left\{\left(M_1^2 - 1\right)\tan^2(\beta) - 1\right\}}{\left[1 + \frac{\gamma + 1}{2} M_1^2\right]\tan(\beta) + \left[1 + \frac{\gamma - 1}{2} M_1^2\right]\tan^3(\beta)}$$

- Regrouping in terms of powers of *tan(* β *)*

$$\left\{\left[1 + \frac{\gamma - 1}{2} M_1^2\right]\tan(\theta)\right\}\tan^3(\beta) - \left(M_1^2 - 1\right)\tan^2(\beta) + \left\{\left[1 + \frac{\gamma + 1}{2} M_1^2\right]\tan(\theta)\right\}\tan(\beta) + 1 = 0$$

Solving for Oblique Shock Wave Angle in Terms of Wedge Angle (improved solution)

(cont'd)

- Letting

$$a = \left\{\left[1 + \frac{\gamma - 1}{2} M_1^2\right]\tan(\theta)\right\}$$

$$b = \left(M_1^2 - 1\right)$$

$$c = \left\{\left[1 + \frac{\gamma + 1}{2} M_1^2\right]\tan(\theta)\right\}$$

$$x = \tan(\beta)$$

- Polynomial has 3 real roots
  - i) weak shock
  - ii) strong shock
  - iii) meaningless solution ($\beta < 0$)

- Result is a cubic equation of the form

$$ax^3 - bx^2 + cx + 1 = 0$$

Solving for Oblique Shock Wave Angle in Terms of Wedge Angle (improved solution)

(cont'd)

- Numerical Solution of Cubic (Newton's method)

$$ax^3 - bx^2 + cx + 1 \equiv f(x) = 0 \rightarrow$$

$$0 = f(x_j) + \left.\frac{\partial f(x)}{\partial x}\right|_j \left(x_{j+1} - x_j\right) + o(x^2)$$

$$x_{j+1} = x_j - \frac{f(x_j)}{\left.\frac{\partial f(x)}{\partial x}\right|_j} = x_j - \frac{a{x_j}^3 - b{x_j}^2 + cx_j + 1}{3a{x_j}^2 - 2bx_j + c}$$

Solving for Oblique Shock Wave Angle in Terms of Wedge Angle (improved solution) (cont'd)

- Collecting terms

$$x_j - \frac{a x_j^{\,3} - b x_j^{\,2} + c x_j + 1}{3a x_j^{\,2} - 2b x_j + c} =$$

$$\frac{3a x_j^{\,3} - 2b x_j^{\,2} + c x_j - \left(a x_j^{\,3} - b x_j^{\,2} + c x_j + 1\right)}{3a x_j^{\,2} - 2b x_j + c} = \frac{2a x_j^{\,3} - b x_j^{\,2} - 1}{3a x_j^{\,2} - 2b x_j + c}$$

Solving for Oblique Shock Wave Angle in Terms of Wedge Angle (improved solution)

(cont'd)

- Solution Algorithm (iterate to convergence)

$$x_{j+1} = \frac{2ax_j^3 - bx_j^2 - 1}{3ax_j^2 - 2bx_j + c}$$

- Where again

$$a = \left\{\left[1 + \frac{\gamma - 1}{2} M_1^2\right] \tan(\theta)\right\}$$

$$b = \left(M_1^2 - 1\right)$$

$$c = \left\{\left[1 + \frac{\gamma + 1}{2} M_1^2\right] \tan(\theta)\right\}$$

$$x = \tan(\beta)$$

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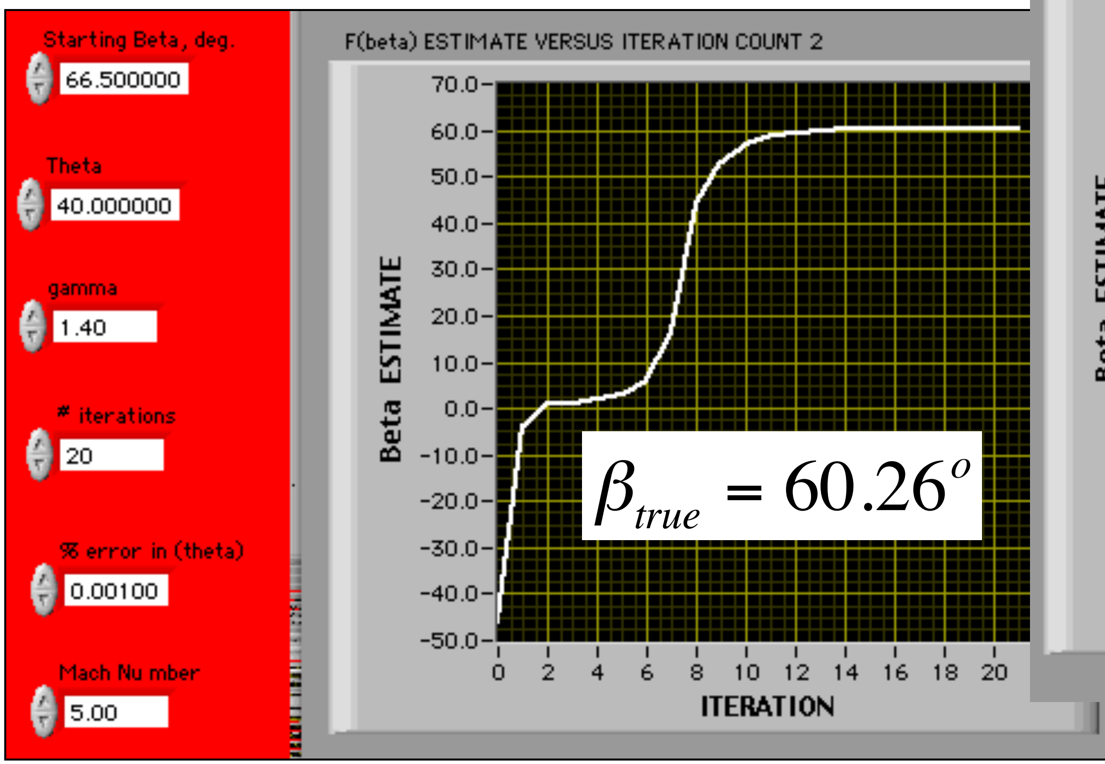
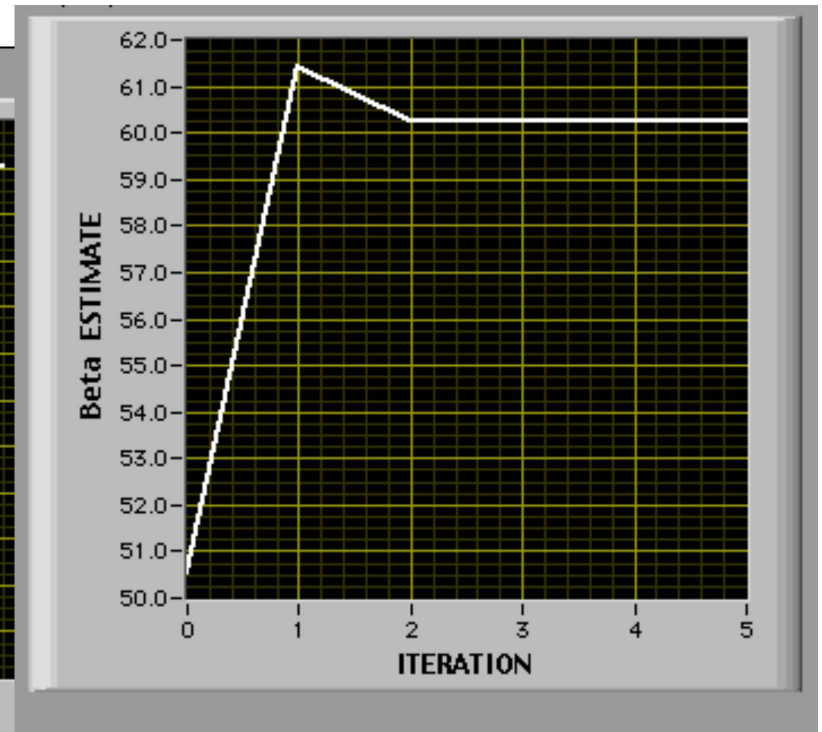
Solving for Oblique Shock Wave Angle in Terms of Wedge Angle (improved solution)

(cont'd)

- Properties of Solver algorithm are much improved



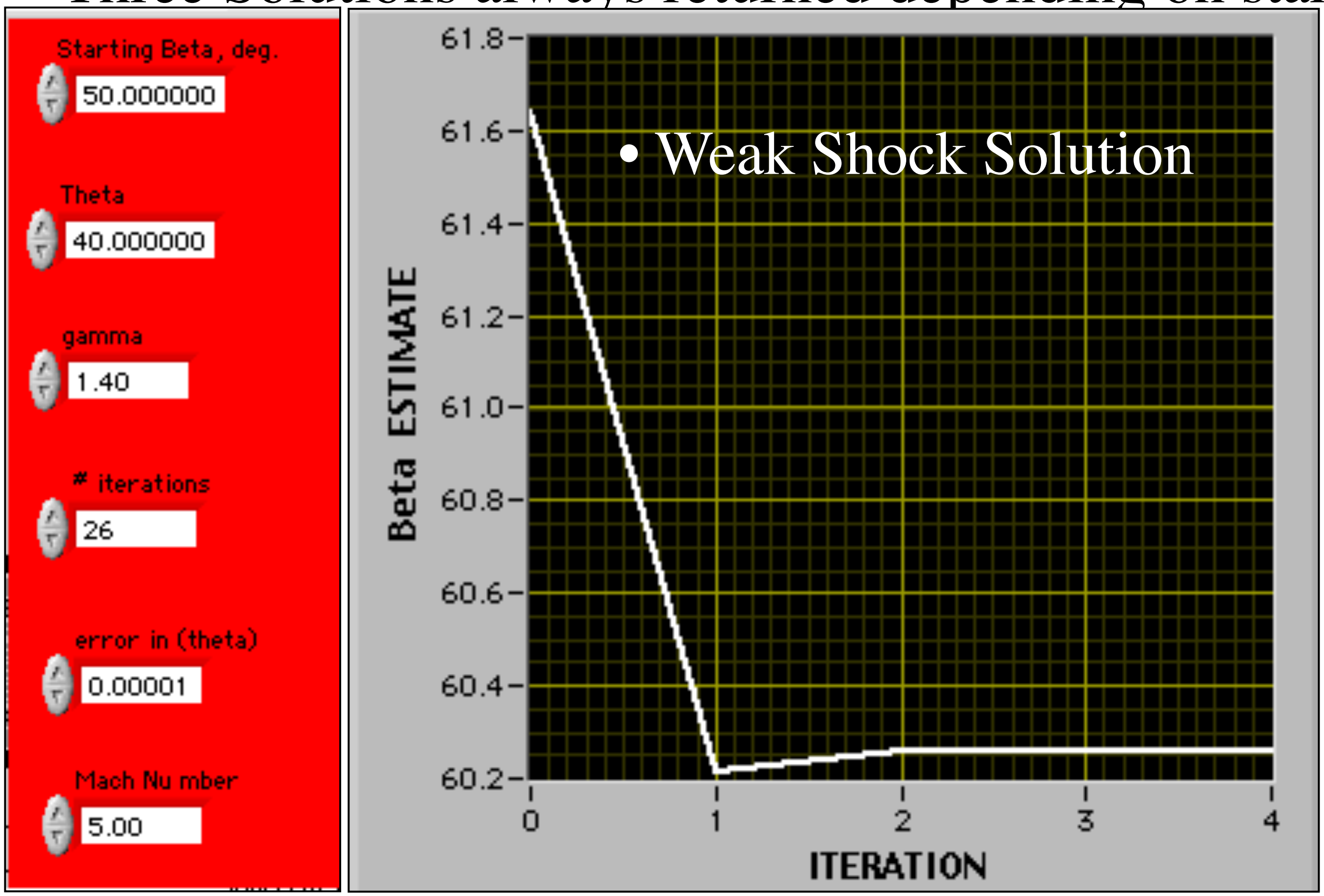
- Original algorithm



- Improved algorithm

Solving for Oblique Shock Wave Angle in Terms of Wedge Angle (improved solution) (cont'd)

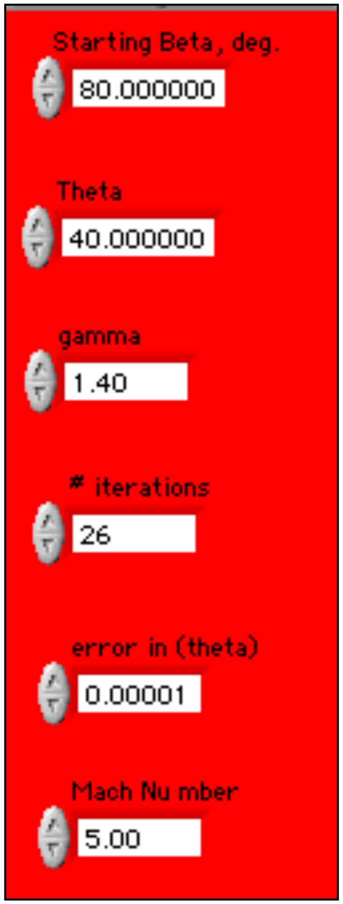
- Three Solutions always returned depending on start condition

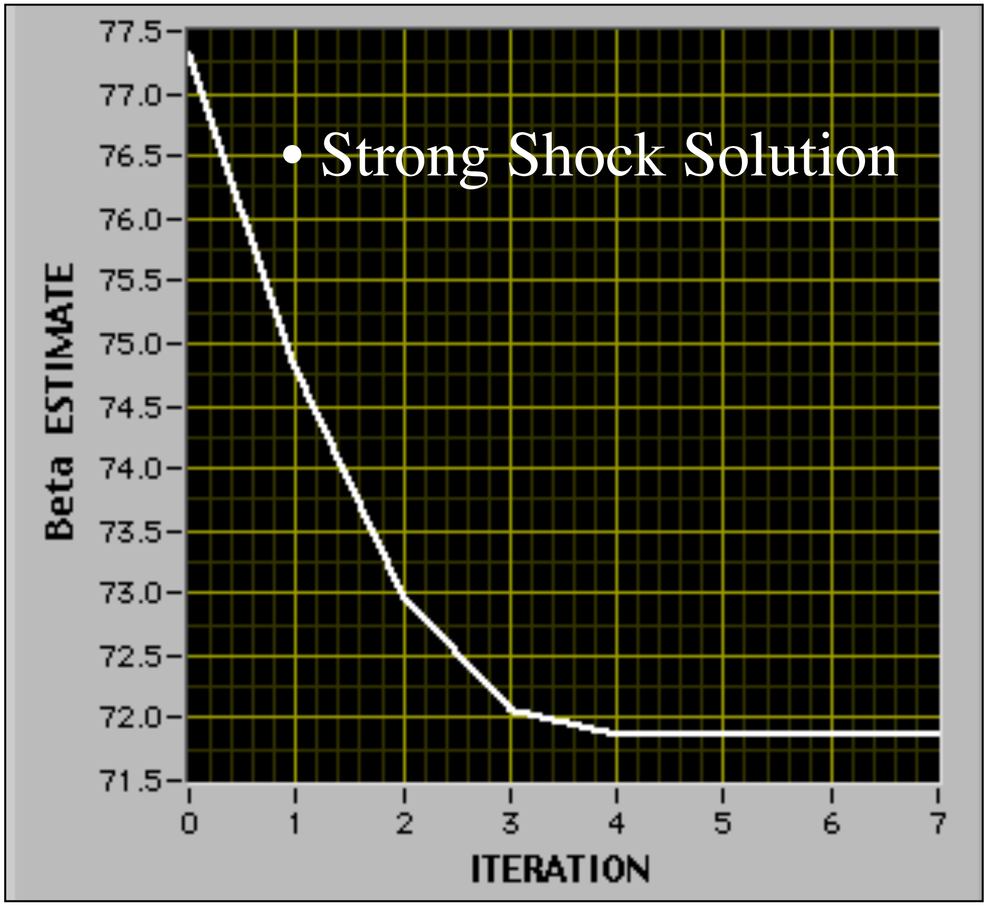


$$\beta_{true} = 60.26^o$$

Solving for Oblique Shock Wave Angle in Terms of Wedge Angle (improved solution) (cont'd)

- Three Solutions always returned depending on start condition

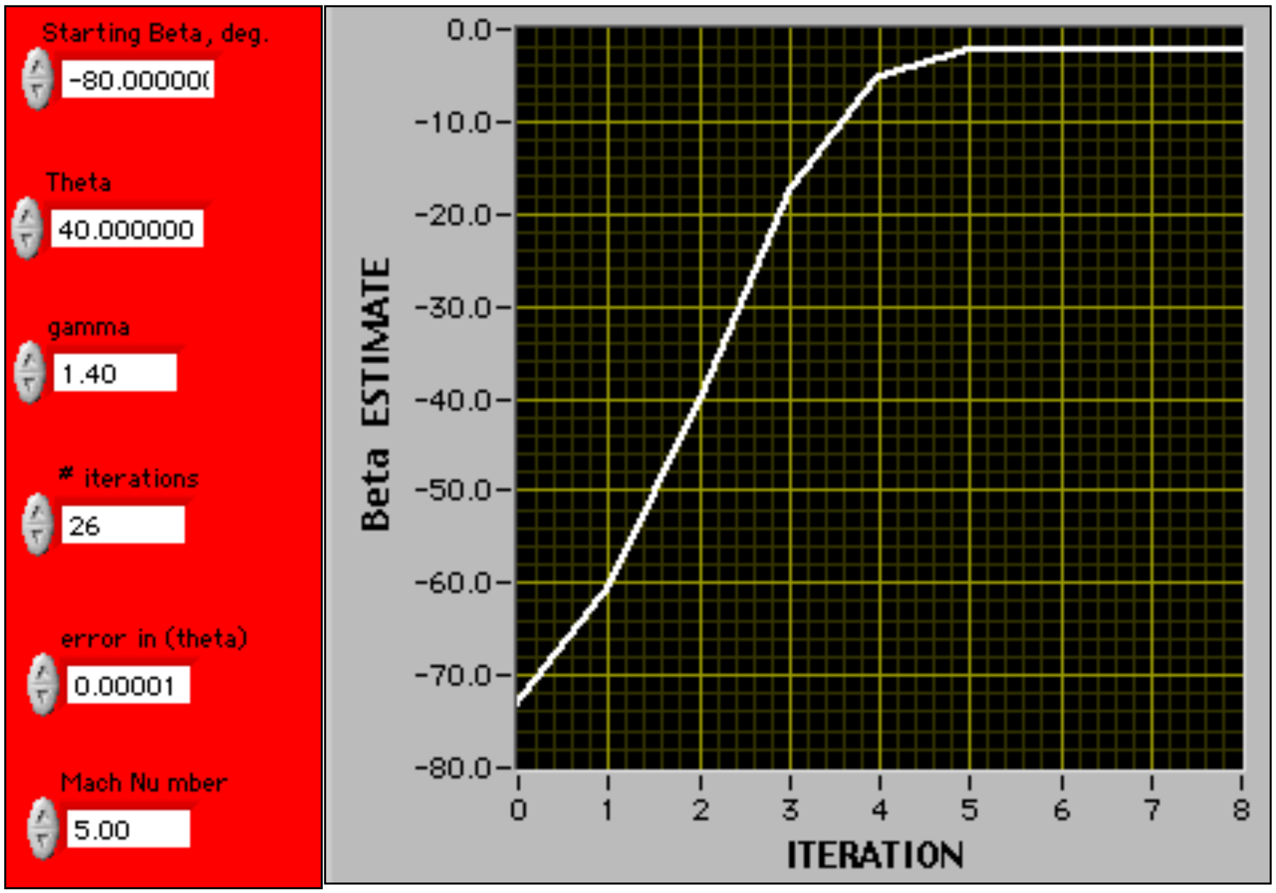




$\beta_{strong} = 71.87^{o}$

Solving for Oblique Shock Wave Angle in Terms of Wedge Angle (improved solution) (cont'd)

- Three Solutions always returned depending on start condition



$$\beta_{meaningless} < 0^{o}$$

Solving for Oblique Shock Wave Angle in Terms of Wedge Angle (explicit solution)

• Explicit Solution … Using guidance from numerical algorithm, can we find Explicit (non -iterative) solution for shock angle?

• Cubic equation has three explicit solutions

i) weak shock

ii) Strong shock

iii) non-physical solution

Solving for Oblique Shock Wave Angle in Terms of Wedge Angle (explicit solution)

- Explicit Solution … Using guidance from numerical algorithm, can we find Explicit (non -iterative) solution for shock angle?

- Root 1: tan[β]=

$$\frac{1}{6a}\left(-2b+\frac{2\,2^{1/3}\,(b^2-3ac)}{\left(-27a^2-2b^3+9abc+\sqrt{-4(b^2-3ac)^3+(27a^2+2b^3-9abc)^2}\right)^{1/3}}+2^{2/3}\left(-27a^2-2b^3+9abc+\sqrt{-4(b^2-3ac)^3+(27a^2+2b^3-9abc)^2}\right)^{1/3}\right)$$

- Root 2: tan[β]=

$$\frac{1}{12a}\left(-4b-\frac{2\,i\,2^{1/3}\left(-i+\sqrt{3}\right)(b^2-3ac)}{\left(-27a^2-2b^3+9abc+\sqrt{-4(b^2-3ac)^3+(27a^2+2b^3-9abc)^2}\right)^{1/3}}+i\,2^{2/3}\left(i+\sqrt{3}\right)\left(-27a^2-2b^3+9abc+\sqrt{-4(b^2-3ac)^3+(27a^2+2b^3-9abc)^2}\right)^{1/3}\right)$$

- Root 3: tan[β]=

$$-\frac{1}{12a}\left(4b+\frac{2\,2^{1/3}\left(1-i\sqrt{3}\right)(b^2-3ac)}{\left(-27a^2-2b^3+9abc+\sqrt{-4(b^2-3ac)^3+(27a^2+2b^3-9abc)^2}\right)^{1/3}}+2^{2/3}\left(1+i\sqrt{3}\right)\left(-27a^2-2b^3+9abc+\sqrt{-4(b^2-3ac)^3+(27a^2+2b^3-9abc)^2}\right)^{1/3}\right)$$

- Break solutions down into manageable form

Solving for Oblique Shock Wave Angle in Terms of Wedge Angle (explicit solution)

• Explicit Solution (From Anderson, pp. 142,143) …

$$\left\{\left[1+\frac{\gamma-1}{2}M_1^2\right]\tan(\theta)\right\}\tan^3(\beta)-\left(M_1^2-1\right)\tan^2(\beta)+\left\{\left[1+\frac{\gamma+1}{2}M_1^2\right]\tan(\theta)\right\}\tan(\beta)+1=0$$

$$\tan(\beta)=\frac{\left(M_1^2-1\right)+2\lambda\cos\left(\frac{4\pi\delta+\cos^{-1}(\chi)}{3}\right)}{3\left[1+\frac{\gamma-1}{2}M_1^2\right]\tan(\theta)}$$

δ = 0 ---> Strong Shock

δ = 1 ---> Weak Shock

$$\lambda=\sqrt{\left(M_1^2-1\right)^2-3\left[1+\frac{\gamma-1}{2}M_1^2\right]\left[1+\frac{\gamma+1}{2}M_1^2\right]\tan^2(\theta)}$$

$$\chi=\frac{\left(M_1^2-1\right)^3-9\left[1+\frac{\gamma-1}{2}M_1^2\right]\left[1+\frac{\gamma-1}{2}M_1^2+\frac{\gamma+1}{4}M_1^4\right]\tan^2(\theta)}{\lambda^3}$$

Solving for Oblique Shock Wave Angle in Terms of Wedge Angle (explicit solution)

- Explicit Solution Check … let $\{M=5,\ \gamma = 1.4\ \ \theta\ ,\ =40°\}$

$$\lambda = \sqrt{\left(M_1^{\,2} - 1\right)^2 - 3\left[1 + \frac{\gamma - 1}{2} M_1^{\,2}\right]\left[1 + \frac{\gamma + 1}{2} M_1^{\,2}\right]\tan^2(\theta)} =$$

$$\left((5^2 - 1)^2 - 3\left(1 + \frac{1.4 - 1}{2} 5^2\right)\left(1 + \frac{1.4 + 1}{2} 5^2\right)\left(\tan\left(\frac{\pi}{180} 40\right)\right)^2\right)^{0.5}$$

$$= 13.5321$$

Solving for Oblique Shock Wave Angle in Terms of Wedge Angle (explicit solution)

- Explicit Solution Check … let *{M=5,* $\gamma = 1.4$ θ , *=40°}*

$$\chi = \frac{\left(M_1^2 - 1\right)^3 - 9\left[1 + \frac{\gamma - 1}{2} M_1^2\right]\left[1 + \frac{\gamma - 1}{2} M_1^2 + \frac{\gamma + 1}{4} M_1^4\right]\tan^2(\theta)}{\lambda^3} =$$

$$\frac{(5^2 - 1)^3 - 9\left(1 + \frac{1.4 - 1}{2} 5^2\right)\left(1 + \frac{1.4 - 1}{2} 5^2 + \frac{1.4 + 1}{4} 5^4\right)\left(\tan\left(\frac{\pi}{180} 40\right)\right)^2}{13.5321^3}$$

$$= -0.267118$$

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Solving for Oblique Shock Wave Angle in Terms of Wedge Angle (explicit solution)

• Explicit Solution Check … let {*M=5*, $\gamma = 1.4$ θ =40°}

δ = 1 ---> Weak Shock

$$\tan(\beta) = \frac{\left(M_1^{\,2}-1\right)+2\lambda\cos\left(\frac{4\pi\delta+\cos^{-1}(\chi)}{3}\right)}{3\left[1+\frac{\gamma-1}{2}M_1^{\,2}\right]\tan(\theta)} =$$

$$\frac{180}{\pi}\,\mathrm{atan}\left(\frac{(5^2-1)+2(13.5321)\cos\left(\frac{4\pi(1)+\mathrm{acos}(-0.26712)}{3}\right)}{3\left(1+\frac{1.4-1}{2}5^2\right)\tan\left(\frac{\pi}{180}40\right)}\right)$$

$= 60.259°$ Check!

Solving for Oblique Shock Wave Angle in Terms of Wedge Angle (explicit solution)

(concluded)

- Explicit Solution Check … let $\{M=5,\ \gamma = 1.4,\ \theta = 40°\}$, $\delta = 0 \rightarrow$ Strong Shock

$$\tan(\beta) = \frac{\left(M_1^2 - 1\right) + 2\lambda\cos\left(\frac{4\pi\delta + \cos^{-1}(\chi)}{3}\right)}{3\left[1 + \frac{\gamma - 1}{2}M_1^2\right]\tan(\theta)} =$$

$$\frac{180}{\pi}\,\mathrm{atan}\left(\frac{(5^2 - 1) + 2\,(13.5321)\,\cos\left(\frac{4\pi\,(0) + \mathrm{acos}\,(-0.26712)}{3}\right)}{3\left(1 + \frac{1.4 - 1}{2}5^2\right)\tan\left(\frac{\pi}{180}40\right)}\right)$$

$$= 71.869°$$

Check!

… OK .. This works …. But is it … the *best method?*

Floating Point Operation (FLOP) Estimate

Input data

Starting Beta, deg.
40.00000

Theta
20.00000

gamma
1.40

iterations
15

% error in (theta)
0.0010(

Mach Nu mber
3.00

$$\tan(\beta) = \frac{(M_1^2 - 1) + 2\lambda \cos\left(\frac{4\pi\delta + \cos^{-1}(\chi)}{3}\right)}{3\left[1 + \frac{\gamma - 1}{2} M_1^2\right]\tan(\theta)}$$

$$\tan(\theta) = \frac{\{(M_1^2 - 1)\tan^2(\beta) - 1\}}{\left[1 + \frac{\gamma + 1}{2} M_1^2\right]\tan(\beta) + \left[1 + \frac{\gamma - 1}{2} M_1^2\right]\tan^3(\beta)}$$

$$x_{j+1} = \frac{2ax_j^3 - bx_j^2 - 1}{3ax_j^2 - 2bx_j + c}$$

# of trials
25000

Total Elapsed time 2

Total elapsed time, exact sec
5.6714783

Total elapsed time, N1 sec 2
60.00535

Total elapsed time, N2 sec 3
5.6519165

Mean Elapsed time

Mean elapsed time, exact sec
0.00022685

Mean elapsed time, N1 sec 2
0.00276213

Mean elapsed time, N2 sec 3
0.00022607

Flop Estimate

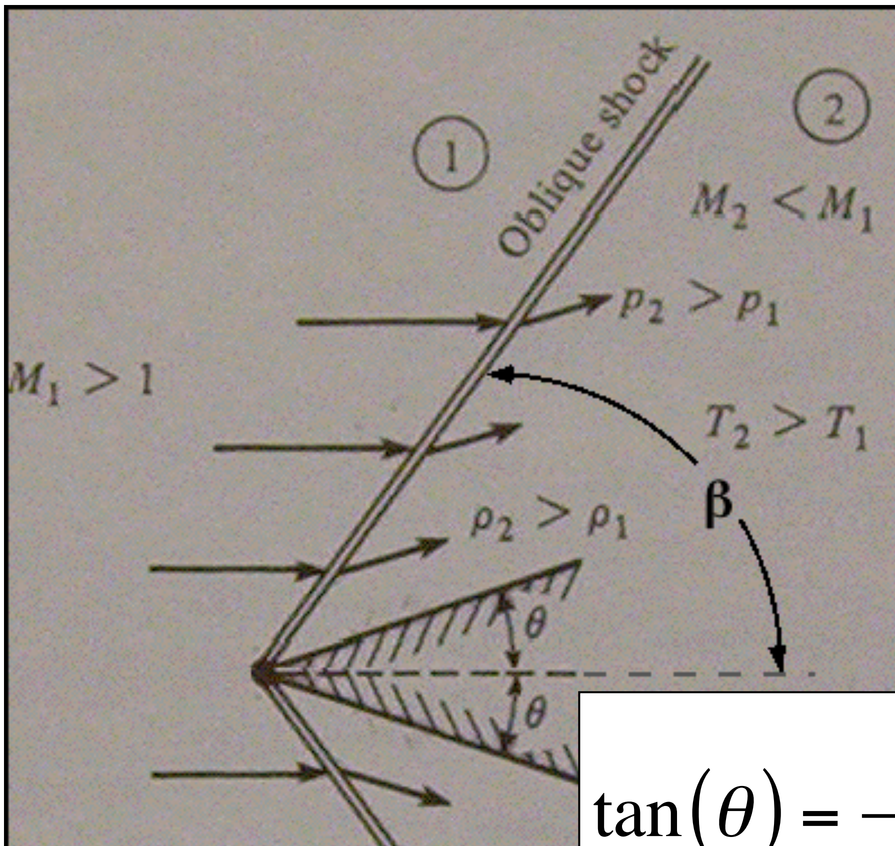
exact
8644

N1
90820

N2
8614

• Actually the simplified numerical Algorithm is slightly faster than closed Form solution

Oblique Shock Waves: Collected Algorithm



- **Properties across Oblique Shock wave ~** $f(M_1, \beta \quad)$
- θ **is the geometric angle that "forces" the flow**

$$\tan(\theta) = \frac{2\left\{M_1^{\,2}\sin^2(\beta) - 1\right\}}{\tan(\beta)\left[2 + M_1^{\,2}\left[\gamma + \cos(2\beta)\right]\right]}$$

Oblique Shock Waves: Collected Algorithm (cont'd)

- **Can be re-written as third order polynomial in tan(θ)**

$$\left\{\left[1+\frac{\gamma-1}{2}M_1^{\,2}\right]\tan(\theta)\right\}\tan^3(\beta)-\left(M_1^{\,2}-1\right)\tan^2(\beta)+\left\{\left[1+\frac{\gamma+1}{2}M_1^{\,2}\right]\tan(\theta)\right\}\tan(\beta)+1=0$$

- **"*Very Easy*" numerical solution**
  - Cubic equation has three solutions
    - i) weak shock
    - ii) Strong shock
    - iii) non-physical solution

$$x_{j+1}=\frac{2ax_j^{\,3}-bx_j^{\,2}-1}{3ax_j^{\,2}-2bx_j+c}$$

$$a=\left\{\left[1+\frac{\gamma-1}{2}M_1^{\,2}\right]\tan(\theta)\right\}$$

$$b=\left(M_1^{\,2}-1\right)$$

$$c=\left\{\left[1+\frac{\gamma+1}{2}M_1^{\,2}\right]\tan(\theta)\right\}$$

$$x=\tan(\beta)$$

Oblique Shock Waves: Collected Algorithm (cont'd)

• **"Less Obvious" explicit solution**

$$\tan(\beta) = \frac{\left(M_1^{\,2} - 1\right) + 2\lambda \cos\left(\dfrac{4\pi\delta + \cos^{-1}(\chi)}{3}\right)}{3\left[1 + \dfrac{\gamma - 1}{2} M_1^{\,2}\right]\tan(\theta)}$$

δ = 0 ---> Strong Shock

δ = 1 ---> Weak Shock

$$\lambda = \sqrt{\left(M_1^{\,2} - 1\right)^2 - 3\left[1 + \frac{\gamma - 1}{2} M_1^{\,2}\right]\left[1 + \frac{\gamma + 1}{2} M_1^{\,2}\right]\tan^2(\theta)}$$

$$\chi = \frac{\left(M_1^{\,2} - 1\right)^3 - 9\left[1 + \frac{\gamma - 1}{2} M_1^{\,2}\right]\left[1 + \frac{\gamma - 1}{2} M_1^{\,2} + \frac{\gamma + 1}{4} M_1^{\,4}\right]\tan^2(\theta)}{\lambda^3}$$

• Either solution
Method is acceptable
For large scale-calculations

Oblique Shock Waves: Collected Algorithm (cont'd)

- **... and the rest of the story …**

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)(M_1 \sin\beta)^2}{\left(2 + (\gamma - 1)(M_1 \sin\beta)^2\right)}$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{(\gamma + 1)}\left((M_1 \sin\beta)^2 - 1\right)$$

$$\frac{T_2}{T_1} = \left[1 + \frac{2\gamma}{(\gamma + 1)}\left((M_1 \sin\beta)^2 - 1\right)\right]\left[\frac{\left(2 + (\gamma - 1)(M_1 \sin\beta)^2\right)}{(\gamma + 1)(M_1 \sin\beta)^2}\right]$$

Oblique Shock Waves: Collected Algorithm (concluded)

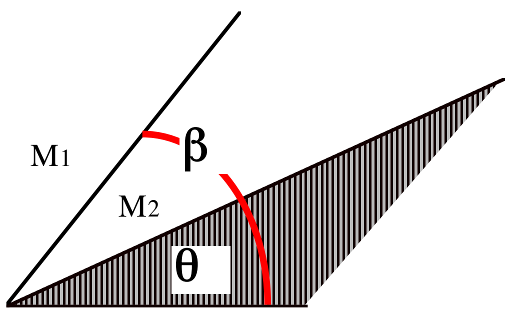
- **... and the rest of the story …**

$$M_{n_2} = \sqrt{\frac{\left(1 + \frac{(\gamma - 1)}{2}(M_1 \sin\beta)^2\right)}{\left(\gamma (M_1 \sin\beta)^2 - \frac{(\gamma - 1)}{2}\right)}} \rightarrow M_2 = \frac{M_{n_2}}{\sin(\beta - \theta)}$$

$$\frac{P_{02}}{P_{01}} = \frac{2}{(\gamma + 1)\left(\gamma (M_1 \sin\beta)^2 - \frac{(\gamma - 1)}{2}\right)^{\frac{1}{\gamma - 1}}} \left(\frac{\left[\frac{(\gamma + 1)}{2}(M_1 \sin\beta)\right]^2}{\left(1 + \frac{\gamma - 1}{2}(M_1 \sin\beta)^2\right)} \right)^{\left(\frac{\gamma}{\gamma - 1}\right)}$$

Example:

• $M_1 = 3.0$, $p_1 = 1\,atm$, T1=288°K, θ =20°, γ =1.4,



• Compute shock wave angle (weak)

• Compute $P0_2$, $T0_2$, p_2, T_2, M_2 … Behind Shockwave

Example: (cont'd)

- $M_1 = 3.0$, $p_1 = 1\,atm$, $\gamma = 1.4$, T1=288°K, $\theta = 20°$
- Explicit Solver for β

$$\lambda = \sqrt{\left(M_1^2 - 1\right)^2 - 3\left[1 + \frac{\gamma - 1}{2} M_1^2\right]\left[1 + \frac{\gamma + 1}{2} M_1^2\right]\tan^2(\theta)} \quad = 7.13226$$

$$\chi = \frac{\left(M_1^2 - 1\right)^3 - 9\left[1 + \frac{\gamma - 1}{2} M_1^2\right]\left[1 + \frac{\gamma - 1}{2} M_1^2 + \frac{\gamma + 1}{4} M_1^4\right]\tan^2(\theta)}{\lambda^3} \quad = 0.93825$$

Example: (cont'd)

- $M_1 = 3.0$, $p_1 = 1\,atm$, $\gamma = 1.4$, T1=288°K, $\theta = 20°$
- $\delta = 1$ (weak shock)

$$\tan(\beta) = \frac{\left(M_1^2 - 1\right) + 2\lambda\cos\left(\frac{4\pi\delta + \cos^{-1}(\chi)}{3}\right)}{3\left[1 + \frac{\gamma - 1}{2}M_1^2\right]\tan(\theta)} \longrightarrow$$

$$\frac{180}{\pi}\,\text{atan}\left(\frac{3^2 - 1 + 2\cdot 7.13226\cos\left(\frac{4\pi(1) + \text{acos}(0.93825)}{3}\right)}{3\left(1 + \frac{1.4 - 1}{2}3^2\right)\tan\left(\frac{\pi}{180}20\right)}\right) =$$

37.764°

Example: (cont'd)

- $M_1 = 3.0$, $p_1 = 1\,atm$, $\gamma = 1.4$, $T1 = 288°K$, $\theta = 20°$

- Compute Normal Component of Free stream mach Number

$$Mn_1 = M_1 \sin\beta = 3\sin\left(\frac{\pi}{180}37.7636\right) = 1.837$$

- Mach "normal" component of number behind shock wave

$$\rightarrow Mn_2 = \sqrt{\frac{\left(1 + \frac{(\gamma - 1)}{2}\left[M_1 \sin(\beta)\right]^2\right)}{\left(\gamma\left[M_1 \sin(\beta)\right]^2 - \frac{(\gamma - 1)}{2}\right)}} = 0.608392$$

Normal Shock Solver

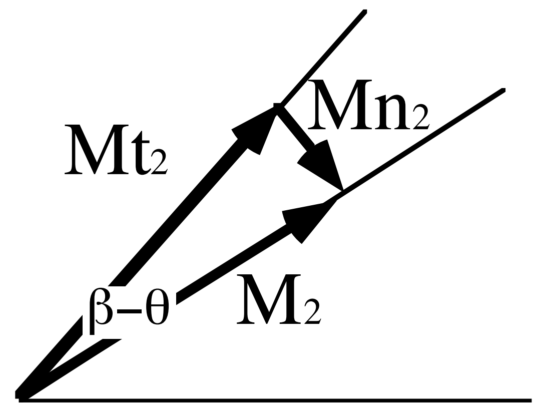
Example: (cont'd)

- $M_1 = 3.0$, $p_1 = 1\,atm$, $\gamma = 1.4$, T1=288°K, $\theta = 20°$

- Mach "normal" component of number behind shock wave

$$\rightarrow Mn_2 = \sqrt{\frac{\left(1 + \frac{(\gamma - 1)}{2}\left[M_1 \sin(\beta)\right]^2\right)}{\left(\gamma\left[M_1 \sin(\beta)\right]^2 - \frac{(\gamma - 1)}{2}\right)}} = 0.608392$$

$$M_2 = \frac{Mn_2}{\sin(\beta - \theta)} = 1.99414$$



FLOW BEHIND SHOCK WAVE IS SUPERSONIC!

Example: (cont'd)

- $M_1 = 3.0$, $p_1 = 1\,atm$, $\gamma = 1.4$, T1=288°K, $\theta = 20°$

- Compute Normal Component of Free stream mach Number

$$Mn_1 = M_1 \sin\beta = 3\sin\left(\frac{\pi}{180}37.7636\right) = 1.837$$

- Compute Pressure ratio across shock

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{(\gamma+1)}\left(Mn_1^{\,2} - 1\right) \longrightarrow \text{Normal Shock Solver}$$

$p_2 = 3.771(1\ atm) = 3.771\ atm$

- Flow is compressed

Example: (cont'd)

- $M_1 = 3.0$, $p_1=1 atm$, $\gamma = 1.4$, T1=288°K, $\theta = 20°$

- Compute Temperature ratio Across Shock

$$\frac{T_2}{T_1} = \left[1 + \frac{2\gamma}{(\gamma+1)}\left(Mn_1^{\,2} - 1\right)\right]\left[\frac{\left(2 + (\gamma - 1) Mn_1^{\,2}\right)}{(\gamma+1) Mn_1^{\,2}}\right]$$

⟶ Normal Shock Solver

$T_2 = 1.5596(288\ °K) = 449.2\ °K$

Example: (cont'd)

- $M_1 = 3.0$, $p_1 = 1\,atm$, $\gamma = 1.4$, T1=288°K, $\theta = 20°$

- Compute Stagnation Pressure ratio across shock

$$Mn_1 = M_1 \sin\beta = 3\sin\left(\frac{\pi}{180}37.7636\right) = 1.837$$

→ Normal Shock Solver

$$\frac{P_{02}}{P_{01}} = 0.7961$$

UtahState UNIVERSITY

Mechanical & Aerospace Engineering

Example: (cont'd)

- $M_1 = 3.0$, $p_1 = 1\,atm$, $\gamma = 1.4$, $T1 = 288°K$, $\theta = 20°$

- Compute Stagnation Pressure ratio (alternate method)

$$\frac{P_{02}}{P_{01}} = \frac{2}{(\gamma+1)\left(\gamma {Mn_1}^2 - \frac{(\gamma-1)}{2}\right)^{\frac{1}{\gamma-1}}}\left(\frac{\left[\frac{(\gamma+1)}{2}Mn_1\right]^2}{\left(1+\frac{\gamma-1}{2}{Mn_1}^2\right)}\right)^{\left(\frac{\gamma}{\gamma-1}\right)}$$

$$\frac{2}{(1.4+1)\left(1.4\left(3\sin\left(\frac{\pi}{180}37.7636\right)\right)^2 - \left(\frac{(1.4-1)}{2}\right)\right)^{\frac{1}{1.4-1}}}\left(\frac{\left(\left(\frac{(1.4+1)}{2}\right)^2\left(3\sin\left(\frac{\pi}{180}37.7636\right)\right)^2\right)}{\left(1+\left(\frac{(1.4-1)}{2}\right)\left(3\sin\left(\frac{\pi}{180}37.7636\right)\right)^2\right)}\right)^{\frac{1.4}{(1.4-1)}}$$

$= 0.7961$

Example: (cont'd)

- $M_1 = 3.0$, $p_1 = 1 atm$, $\gamma = 1.4$, T1=288°K, $\theta = 20°$

- Compute Stagnation Pressure

$$P_{02} = \frac{P_{02}}{P_{01}} \times \frac{P_{01}}{p_1} \times p_1 = \frac{P_{02}}{P_{01}} \times \left[1 + \frac{\gamma - 1}{2} M_1^{\ 2}\right]^{\frac{\gamma}{\gamma - 1}} \times p_1$$

$$(0.7961)\left[1 + \frac{1.4 - 1}{2} 3^2\right]^{\frac{1.4}{1.4-1}} \times 1 atm =$$

=29.24 atm

Example: (cont'd)

- $M_1 = 3.0$, $p_1=1 atm$, $\gamma = 1.4$, T1=288°K, $\theta = 20°$

- Compute Stagnation Temperature behind shock

$$T_{02} = T_{01} = \frac{T_{01}}{T_1} \times T_1 = \left[1 + \frac{\gamma - 1}{2} M_1{}^2\right] \times T_1$$

$$\left[1 + \frac{1.4 - 1}{2} 3^2\right] \times 288^o K =$$

=806.4 °K

Example: (summary)

Flow is supersonic
Behind shock wave

Ahead of shock	*Behind Shock*
$M_\infty = 3.0$	$M_2 = 1.99414$
$\theta = 20^0$	$\beta = 37.764^o$
$p_\infty = 1\ atm$	$p_2 = 3.771\ atm$
$T_\infty = 288^o K$	$T_2 = 449.2^o K$
$P_{0_\infty} = 36.73\ atm$	$P_{0_2} = 29.24\ atm$
$T_{0_\theta} = 806.4^o K$	$T_{0_2} = 806.4^o K$
$M_{1_n} = 1.837$	$M_{2_n} = 0.608392$
$M_{1_t} = 2.372$	$M_{2_t} = 1.21333$

What Happens When ….

θ = 34.01° ?

Select Weak Shock Wave Solution

Flow is ***subsonic***
Behind shock wave

$$\beta = 63.786^o \rightarrow M_{1_n} = 3 \cdot \sin(\frac{\pi}{180} 63.786^o) = 2.69145 \rightarrow M_{2_n} = 0.49631$$

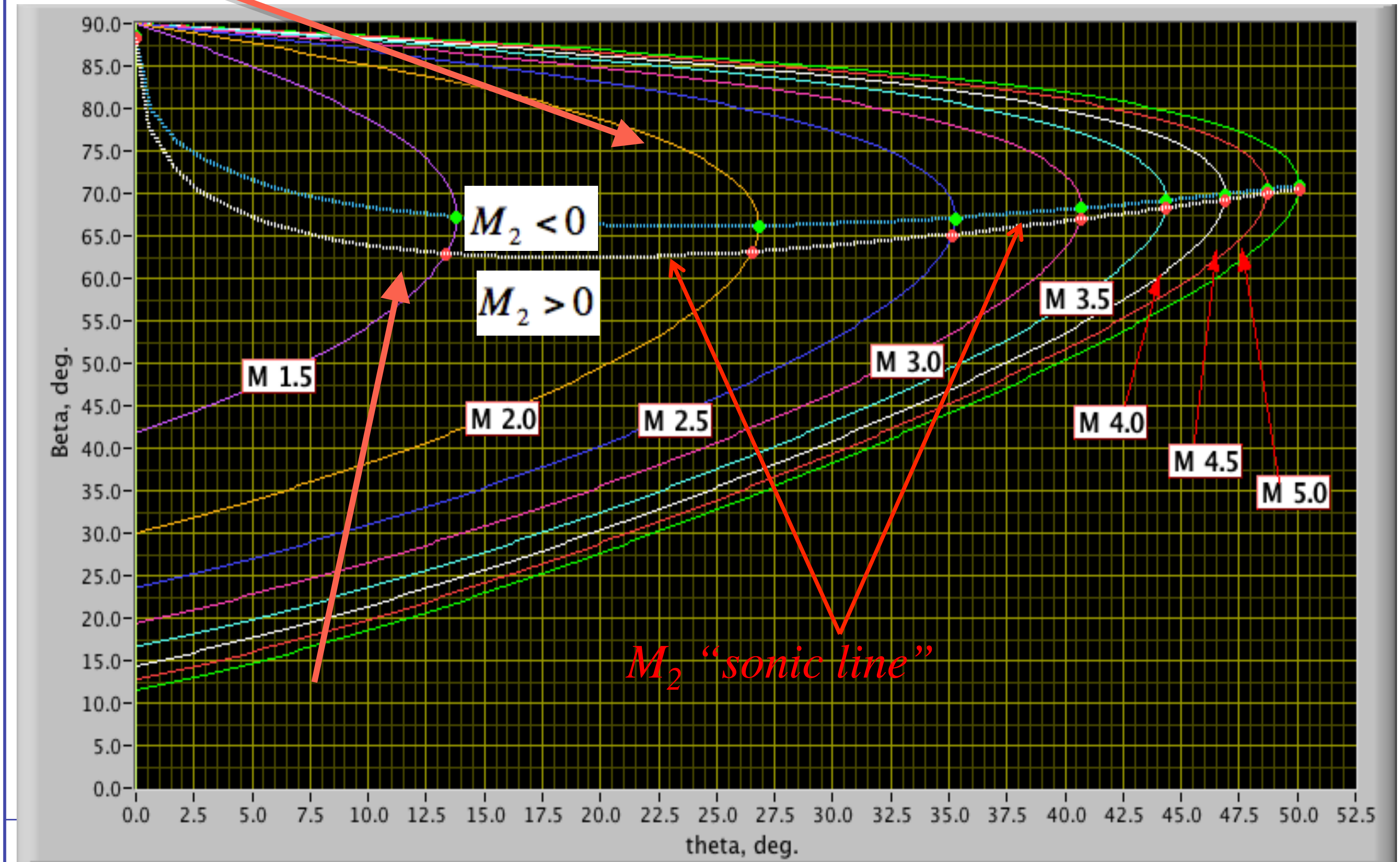
$$M_2 = \frac{M_{2_n}}{\sin(\beta - \theta)} = \frac{0.49631}{\sin(\frac{\pi}{180} 63.786^o - 34.01^o)} = 0.999394$$

Select Strong Shock Wave Solution

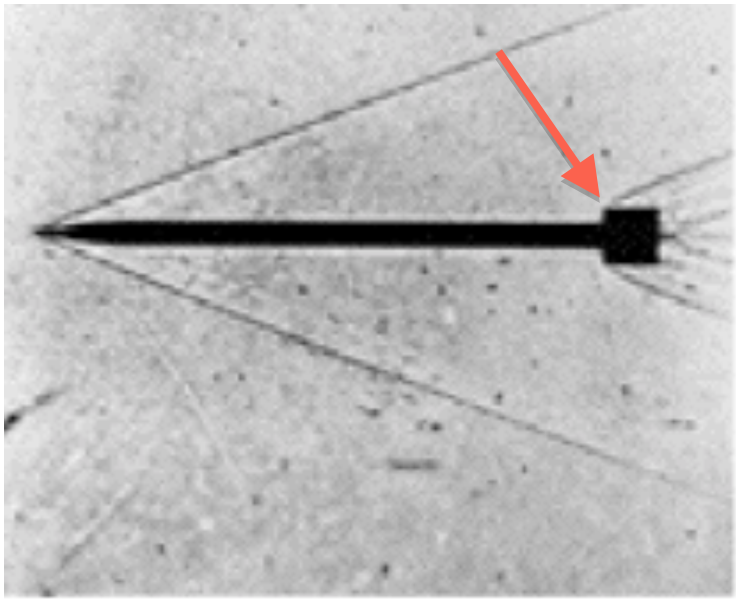
$$\beta = 66.6448^o \rightarrow M_{1_n} = 3 \cdot \sin(\frac{\pi}{180} 63.786^o) = 2.75419 \rightarrow M_{2_n} = 0.491498$$

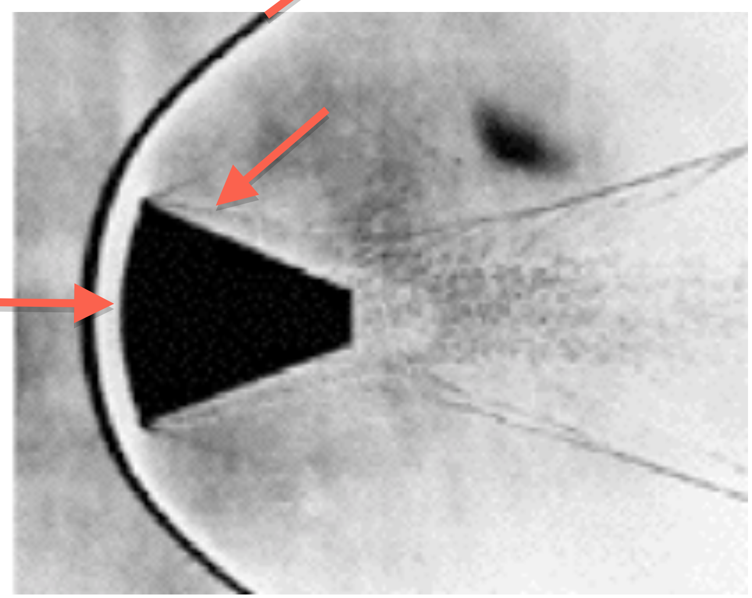
$$M_2 = \frac{M_{2_n}}{\sin(\beta - \theta)} = \frac{0.491498}{\sin(\frac{\pi}{180} 63.786^o - 34.01^o)} = 0.989705$$

Add another Curve to β−θ−M diagram….



Weak, Strong, and Detached Shockwaves





What Happens when …

- $M_1 = 3.0$, $p_1 = 1\,atm$, $\gamma = 1.4$, T1=288°K, $\theta = 0.00001°$

- Explicit Solver for β

$$\lambda = \sqrt{\left(M_1^2 - 1\right)^2 - 3\left[1 + \frac{\gamma - 1}{2} M_1^2\right]\left[1 + \frac{\gamma + 1}{2} M_1^2\right]\tan^2(\theta)} \quad = 8.0$$

$$\chi = \frac{\left(M_1^2 - 1\right)^3 - 9\left[1 + \frac{\gamma - 1}{2} M_1^2\right]\left[1 + \frac{\gamma - 1}{2} M_1^2 + \frac{\gamma + 1}{4} M_1^4\right]\tan^2(\theta)}{\lambda^3} \quad = 1.0$$

What Happens when (cont'd)

•M_1 = 3.0, p_1=1atm, γ =1.4, T1=288°K, θ =0.00001°

$$\tan(\beta) = \frac{\left(M_1^2 - 1\right) + 2\lambda\cos\left(\frac{4\pi\delta + \cos^{-1}(\chi)}{3}\right)}{3\left[1 + \frac{\gamma - 1}{2}M_1^2\right]\tan(\theta)} \longrightarrow$$

β =19.47° $\longrightarrow$ $\mu = \frac{180}{\pi}\sin^{-1}\left[\frac{1}{M_1}\right] = 19.47^o$

• "mach line"

What Happens when (cont'd)

- $M_1 = 3.0$, $p_1 = 1\,atm$, $\gamma = 1.4$, T1=288°K, $\beta = 0.00001°$
- Χομπυτε Normal Component of Free stream mach Number

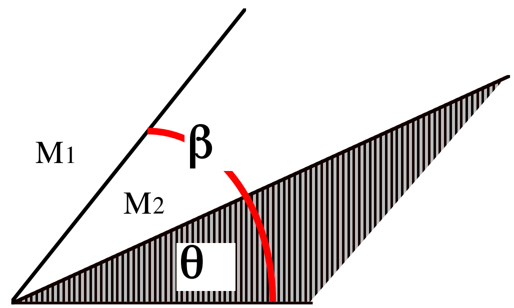
$$Mn_1 = M_1 \sin\beta = \quad 1.0000$$

- $$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{(\gamma+1)}\left(Mn_1^{\ 2} - 1\right) \quad = 1.0$$ **(NO COMPRESSION!)**

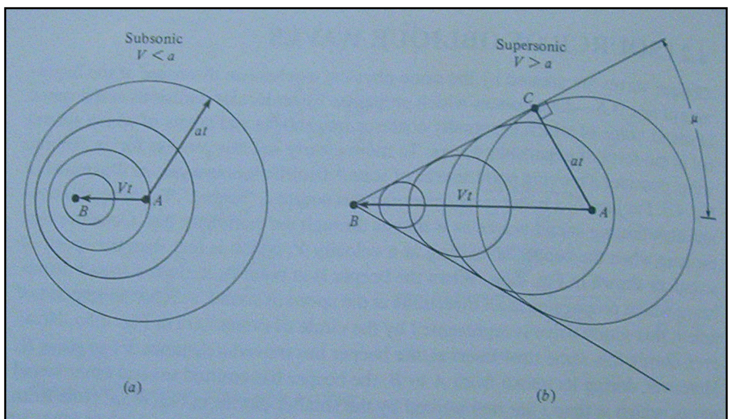
Expansion Waves

- So if

θ >0 .. Compression around corner



θ =0 … no compression across shock

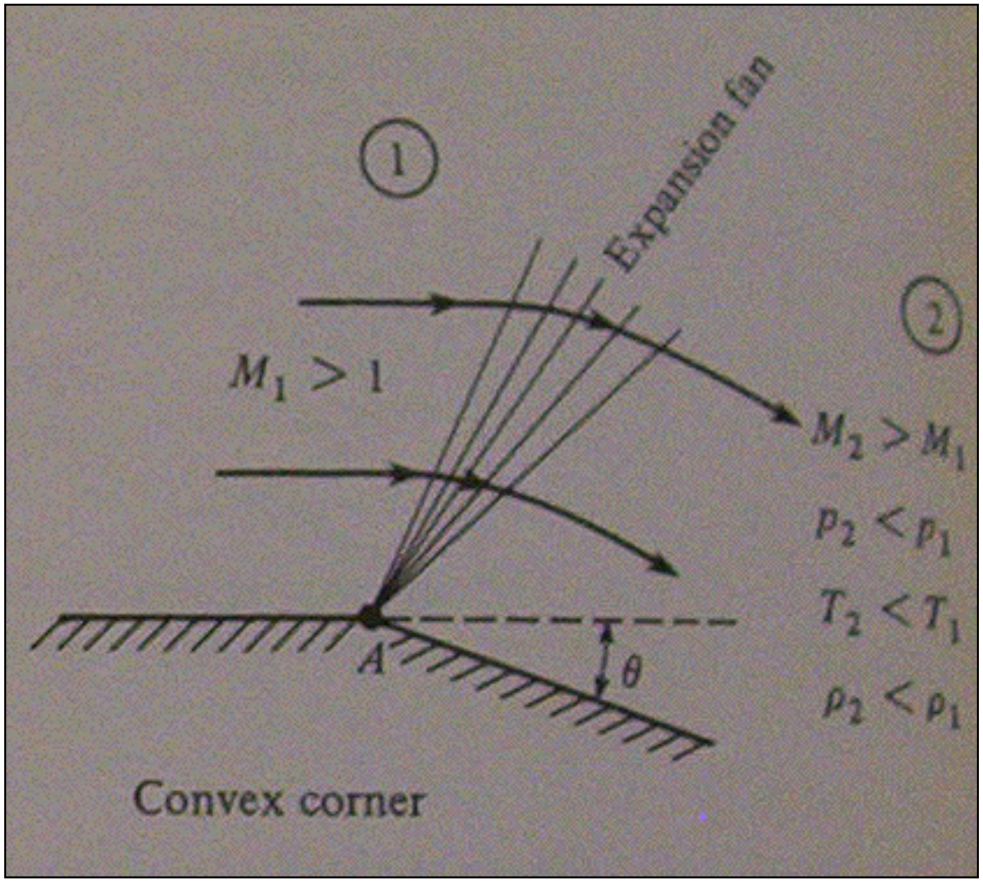


The propagation of disturbances in (a) subsonic and (b) supersonic flow.

Expansion Waves (concluded)

- Then it follows that

  θ <0 .. We get an *expansion wave*



- Next

Prandtl-Meyer
Expansion waves