Marcin Bielecki, Advanced Macroeconomics

1 Ramsey-Cass-Koopmans model

The "workhorse model" of modern macroeconomics.

Authors: **Frank Ramsey (1928)**, **David Cass (1965)**, **Tjalling Koopmans (1965)**

Assumptions similar to the Solow-Swan model:

- Closed economy
- No government (for now)
- Single, homogenous final good with its price normalized to 1 in each period (all variables are expressed in real terms)
- All households supply a unit of labor, number of people equal to number of workers
- Output is produced according to a neoclassical production function
- Two types of representative, optimizing agents:
  - Firms
  - Households
- Households are solving a utility maximizing problem – we have a well defined welfare measure

1.1 Production function

A production function describes how capital K and labor L is transformed into output Y using technology A, and is written in its most general form as $Y_t = F(K_t, L_t, A_t)$.

A neoclassical production function $F : \mathbb{R}^3_+ \to \mathbb{R}_+$ has following properties:

- Continuous and at least twice differentiable
- Constant returns to scale in K and L. Increasing both capital and labor inputs by a certain proportion translates to increase of output produced by that same proportion:

$$F(\gamma K, \gamma L, A) = \gamma F(K, L, A) = \gamma Y \quad \text{for all } \gamma > 0$$

- Positive but diminishing marginal products of K and L:

$$\frac{\partial F(K,L,A)}{\partial K} \equiv F_K(K,L,A) > 0 \qquad \frac{\partial F(K,L,A)}{\partial L} \equiv F_L(K,L,A) > 0$$
$$\frac{\partial^2 F(K,L,A)}{\partial K^2} \equiv F_{KK}(K,L,A) < 0 \qquad \frac{\partial^2 F(K,L,A)}{\partial L^2} \equiv F_{LL}(K,L,A) < 0$$

- **Inada** conditions:

$$\lim_{K\to 0} F_K(K,L,A) = \infty \qquad \lim_{L\to 0} F_L(K,L,A) = \infty$$
$$\lim_{K\to \infty} F_K(K,L,A) = 0 \qquad \lim_{L\to \infty} F_L(K,L,A) = 0$$

The assumption of constant returns to scale has important implications. First, it makes the number of firms indeterminate, as the economy behaves identically in the case of one firm that employs all resources and in the case of M firms, each employing a $1/M$ fraction of all resources. Therefore we often assume that the firms sector is represented by a single representative firm. Second, the economic profits of the firms sector are equal to zero. To show that, we will need to use a mathematical theorem.

Definition
Let $X \in \mathbb{N}$. A function $g : \mathbb{R}^{X+2} \to \mathbb{R}$ is homogenous of degree m in $x \in \mathbb{R}$ and $y \in \mathbb{R}$ if

$$g(\gamma x, \gamma y, z) = \gamma^m g(x, y, z) \quad \text{for all } \gamma \in \mathbb{R}_+ \text{ and } z \in \mathbb{R}^X$$

According to the above definition, a neoclassical production function is homogenous of degreee one in capital K and labor L.

Euler's Homogeneous Function Theorem
Suppose that $g : \mathbb{R}^{X+2} \to \mathbb{R}$ is differentiable in $x \in \mathbb{R}$ and $y \in \mathbb{R}$, with partial derivatives denoted by g_x and g_y, and is homogeneous of degree m in x and y. Then:

$$m\gamma^{m-1} g(x, y, z) = g_x(x, y, z)\, x + g_y(x, y, z)\, y \quad \text{for all } x \in \mathbb{R},\ y \in \mathbb{R} \text{ and } z \in \mathbb{R}^X$$

Moreover, $g_x(x, y, z)$ and $g_x(x, y, z)$ are themselves homogeneous of degree $m-1$ in x and y.

Proof
Differentiate the definition for function homogenous of degree m with respect to γ:

$$m\gamma^{m-1}(x, y, z) = g_x(\gamma x, \gamma y, z)\, x + g_y(\gamma x, \gamma y, z)\, y$$

Setting $\gamma = 1$ proves the first result of the theorem. To obtain the second result, differentiate the definition for function homogenous of degree m with respect to x:

$$g_x(\gamma x, \gamma y, z) \cdot \gamma = \gamma^m g_x(x, y, z)$$
$$g_x(\gamma x, \gamma y, z) = \gamma^{m-1} g_x(x, y, z)$$

In our context, the theorem translates to the following results:

$$F(K, L, A) = F_K(K, L, A) \cdot K + F_L(K, L, A) \cdot N$$

and:

$$F_K(\gamma K, \gamma L, A) = F_K(K, L, A)$$

1.2 Firms

Representative firms produce according to the neoclassical production function:

$$Y_t = F(K_t, L_t, A_t)$$

where Y is real GDP, K is capital stock, A denotes the technology/productivity level and L is employment. Firms want to maximize their profits:

$$\max \quad \Pi_t = F(K_t, L_t, A_t) - w_t L_t - r_t^k K_t$$

where r^k denotes rental cost of capital. The rental cost of capital is related to the real interest rate in the financial market in the following way:

$$r_t = r_t^k - \delta$$

The households are indifferent between allocating assets in the financial market, yielding return r, and owning physical capital, yielding net return of $r^k - \delta$.

Let us rewrite the production function in intensive form (per worker):

$$y_t \equiv \frac{Y_t}{L_t} = \frac{1}{L_t} F(K_t, L_t, A_t) = F\left(\frac{K_t}{L_t}, \frac{L_t}{L_t}, A_t\right) = F(k_t, 1, A_t) \equiv f(k_t)$$

where $y \equiv Y/L$ denotes GDP per worker and $k \equiv K/L$ denotes capital stock per worker.

The profit maximization problem using per worker variables is:

$$\max \quad \Pi_t = L_t \left[f(k_t) - w_t - r_t^k k_t\right]$$

Firms will choose employment level L and capital per worker k given wage w and capital rental cost r^k.

First order conditions:

$$\begin{aligned} k_t &: \quad L_t \left[f'(k_t) - r_t^k\right] = 0 \\ L_t &: \quad f(k_t) - w_t - r_t^k k_t = 0 \end{aligned}$$

Simplify and rewrite:

$$\begin{aligned} r_t^k &= f'(k_t) \\ w_t &= f(k_t) - r_t^k k_t = f(k_t) - f'(k_t) k_t \end{aligned}$$

The real interest rate is given by:

$$r_t = r_t^k - \delta = f'(k_t) - \delta$$

We can verify that due to perfect competition and constant returns to scale economic profits are zero:

$$\Pi = L_t \left[f(k_t) - w_t - r_t^k k_t\right] = L_t \left[f(k_t) - \left[f(k_t) - f'(k_t) k_t\right] - f'(k_t) k_t\right] = 0$$

The above result was guaranteed by the Euler theorem.

1.3 Households

Representative, infinitely lived households solve the following utility maximization problem:

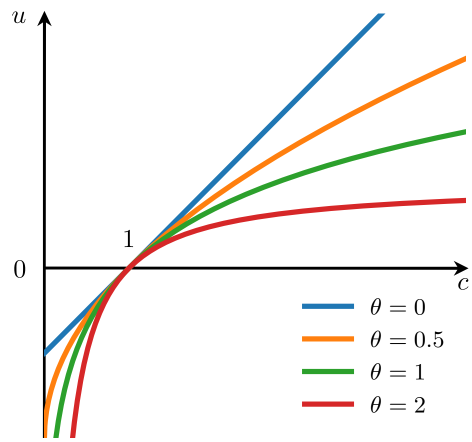
$$\begin{aligned} \max \quad & U_0 = \sum_{t=0}^{\infty} \beta^t N_t \frac{c_t^{1-\theta} - 1}{1-\theta} \\ \text{subject to} \quad & Assets_{t+1} = w_t L_t + (1 + r_t) Assets_t - C_t, \quad t = 0, 1, \ldots, \infty \end{aligned}$$

where β is the discount (impatience) factor, N denotes population, c stands for consumption per capita, $\theta > 0$ is a parameter which we will discuss in a moment, $Assets$ denote total assets of the household sector, r is the real interest rate, w denotes the wage earned by workers and $C = c \cdot N$ is the total consumption of households. We also assume that all agents are working, so that $L_t = N_t$.

The instantaneous utility function, $\left(c^{1-\theta} - 1\right) / (1-\theta)$ is a Constant Relative Risk Aversion (CRRA) function and can be thought of as a generalization of the familiar logarithmic function. One can show using L'Hôpital's rule (H) that as $\theta \to 1$, the CRRA function becomes the logarithmic function:

$$\lim_{\theta \to 1} \frac{c^{1-\theta} - 1}{1-\theta} = \lim_{\theta \to 1} \frac{\exp\left[(1-\theta)\ln c\right] - 1}{1-\theta} = (H) = \lim_{\theta \to 1} \frac{\exp\left[(1-\theta)\ln c\right] \cdot (-\ln c)}{-1} = \ln c$$

The parameter θ measures risk aversion of the agent. In our context this will mean that an agent with high θ will prefer smooth and stable paths of consumption relative to ones that imply large changes in consumption over time. The shape of the CRRA function for various levels of θ can be seen below:



We will assume that the population grows at a constant rate n, that is:

$$N_{t+1} = (1+n)\, N_t \quad \text{and} \quad N_t = (1+n)^t\, N_0$$

Now we will rewrite the original problem in per capita terms. For the utility function it is easy:

$$U_0 = \sum_{t=0}^{\infty} \beta^t (1+n)^t N_0 \frac{c_t^{1-\theta}-1}{1-\theta} = N_0 \sum_{t=0}^{\infty} [\beta (1+n)]^t \frac{c_t^{1-\theta}-1}{1-\theta}$$

Since the utility orderings are invariant with respect to monotonic transformations, we can without loss of generality "forget" about the initial population size N_0 by dividing the original utility function by N_0.

The budget constraint is also easy to reformulate, just be careful about the distinction between periods t and $t+1$:

$$\begin{aligned} Assets_{t+1} &= w_t L_t + (1+r_t)\, Assets_t - C_t \quad | \quad : N_t \\ \frac{Assets_{t+1}}{N_t} &= \frac{w_t L_t}{N_t} + \frac{(1+r_t)\, Assets_t}{N_t} - \frac{C_t}{N_t} \\ \frac{Assets_{t+1}}{N_{t+1}} \frac{N_{t+1}}{N_t} &= w_t + (1+r_t)\, a_t - c_t \\ (1+n)\, a_{t+1} &= w_t + (1+r_t)\, a_t - c_t \end{aligned}$$

where we define per capita assets $a_t \equiv Assets_t / N_t$. Notice how $Assets_{t+1}/N_t \neq a_{t+1}$ and we have to take into account population growth. We are now ready to solve the problem using the Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} [\beta (1+n)]^t \left\{ \frac{c_t^{1-\theta}-1}{1-\theta} + \lambda_t \left[w_t + (1+r_t)\, a_t - c_t - (1+n)\, a_{t+1} \right] \right\}$$

In each time period t the choice variables for this problem are consumption per capita in the current period c_t and assets per capita in the next period a_{t+1}. It may be easier to derive the first order conditions if we expand the Lagrangian first:

$$\mathcal{L} = \ldots + [\beta(1+n)]^t \left\{ \frac{c_t^{1-\theta}-1}{1-\theta} + \lambda_t [w_t + (1+r_t)a_t - c_t - (1+n)a_{t+1}] \right\}$$
$$+ [\beta(1+n)]^{t+1} \left\{ \frac{c_{t+1}^{1-\theta}-1}{1-\theta} + \lambda_{t+1} [w_{t+1} + (1+r_{t+1})a_{t+1} - c_{t+1} - (1+n)a_{t+2}] \right\} + \ldots$$

First order conditions:

$$\begin{aligned} c_t &: \quad [\beta(1+n)]^t \left\{ c_t^{-\theta} + \lambda_t [-1] \right\} = 0 \\ a_{t+1} &: \quad [\beta(1+n)]^t \{\lambda_t [-(1+n)]\} + [\beta(1+n)]^{t+1} \{\lambda_{t+1} [(1+r_{t+1})]\} = 0 \end{aligned}$$

Simplify and rewrite:

$$\begin{aligned} \lambda_t &= c_t^{-\theta} \\ \lambda_t &= \beta\lambda_{t+1}(1+r_{t+1}) \end{aligned}$$

Resulting Euler equation:

$$c_t^{-\theta} = \beta c_{t+1}^{-\theta}(1+r_{t+1})$$

Notice that if $\theta = 1$, the equation can be written in the form we have previously seen for the logarithmic utility function:

$$c_{t+1} = \beta(1+r_{t+1})c_t$$

To gain more intuition, we introduce the discount rate ρ which is related to the discount factor β in the following way:

$$\beta = \frac{1}{1+\rho}$$

The discount rate can be interpreted as a "psychological" interest rate of an agent. The Euler equation now can be written as:

$$\frac{c_{t+1}}{c_t} = \left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta}$$

and it implies that consumption increases over time if the real interest rate exceeds the "psychological" interest rate. In the situation where $r_{t+1} > \rho$, an agent willing to save and it means that she consumes less in the present to consume more in the future. The parameter θ will affect how strongly the difference between real and "psychological" interest rates influences changes in consumption. The higher the θ, the smaller are the changes of consumption in response to the difference between the interest rates.

1.4 General Equilibrium

In a closed economy, the only asset in positive net supply is capital, because all the borrowing and lending must cancel out within the economy. Hence, equilibrium in the asset market requires $a = k$. We will modify the households' budget constraint accordingly and in the next step plug in the expressions for prices:

$$\begin{aligned}
(1+n)\,a_{t+1} &= w_t + (1+r_t)\,a_t - c_t \\
(1+n)\,k_{t+1} &= w_t + (1+r_t)\,k_t - c_t \\
(1+n)\,k_{t+1} &= [f(k_t) - f'(k_t)\,k_t] + (1 + [f'(k_t) - \delta])\,k_t - c_t \\
(1+n)\,k_{t+1} &= f(k_t) + (1-\delta)\,k_t - c_t
\end{aligned}$$

If we "reverse-engineer" this equation into form with aggregate terms by multiplying both sides by N_t, we get:

$$\begin{aligned}
N_t \frac{K_{t+1}}{N_{t+1}} \frac{N_{t+1}}{N_t} &= N_t y_t + (1-\delta)\,N_t k_t - N_t c_t \\
K_{t+1} &= Y_t + (1-\delta)\,K_t - C_t \\
Y_t &= C_t + I_t
\end{aligned}$$

where we used the accounting definition of gross investment, $I_t = K_{t+1} - (1-\delta)\,K_t$. Unsurprisingly, the above equation is a variant of the national accounting identity, $Y = C + I + G + NX$, for a closed economy ($NX = 0$) with no government sector ($G = 0$).

If we go back to the Euler equation, we can replace the real interest rate with the marginal product of capital net of depreciation:

$$\begin{aligned}
c_t^{-\theta} &= \beta c_{t+1}^{-\theta}\,(1 + r_{t+1}) \\
c_t^{-\theta} &= \beta c_{t+1}^{-\theta}\,(1 + f'(k_{t+1}) - \delta)
\end{aligned}$$

The dynamics of the entire economy are summed up by the following two equations:

$$\begin{aligned}
\text{Euler equation} \quad & \left(\frac{c_{t+1}}{c_t}\right)^{\theta} = \beta\,(1 - \delta + f'(k_{t+1})) \\
\text{Resource constraint} \quad & (1+n)\,k_{t+1} = f(k_t) + (1-\delta)\,k_t - c_t
\end{aligned}$$

1.5 Steady state and transition dynamics

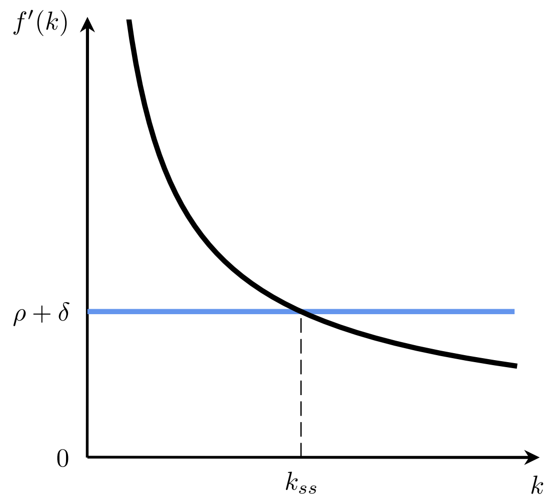
The steady state of an economic system is a situation where all variables grow at constant rates. Often economists distinguish between the steady state proper and balanced growth path situations, where the former exhibits zero growth in income per person, while the latter exhibits constant positive growth.

Without technology improvements, variables per person stabilize over time. We can easily find values of c and k that put the system at rest. Start with the Euler equation:

$$\left(\frac{c_{ss}}{c_{ss}}\right)^{\theta} = \beta\left(1-\delta+f'\left(k_{ss}\right)\right)$$
$$\frac{1}{\beta} = 1-\delta+f'\left(k_{ss}\right)$$
$$f'\left(k_{ss}\right) = \frac{1}{\beta}-(1-\delta)$$

Utilizing our "psychological" interest rate ρ, the Euler equation in the steady state simplifies to:

$$f'\left(k_{ss}\right)=\rho+\delta$$



The steady state level of capital per worker is found by equating the marginal product of capital with the sum of households' discount rate ρ and depreciation rate δ.

If we assume a specific production function, we can obtain an explicit formula for the steady state level of capital per worker. For Cobb-Douglas production function:

$$F(K,L,A) = K^{\alpha}(AN)^{1-\alpha}$$
$$f(k) = k^{\alpha}A^{1-\alpha}$$
$$f'(k) = \alpha k^{\alpha-1}A^{1-\alpha}$$

$$\alpha k_{ss}^{\alpha-1}A^{1-\alpha} = \rho+\delta$$
$$k_{ss} = \left(\frac{\rho+\delta}{\alpha A^{1-\alpha}}\right)^{\frac{1}{\alpha-1}} = A\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}}$$

If we go back to the Euler equation in the form relating real interest rate and discount rate of households:

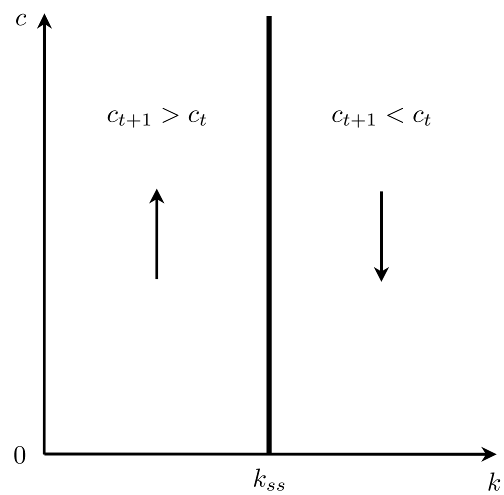
$$\frac{c_{t+1}}{c_t} = \left(\frac{1 + r_{t+1}}{1 + \rho}\right)^{1/\theta}$$

we can see that consumption will increase over time whenever $r > \rho$:

$$r = r^k - \delta = f'(k) - \delta$$

$$r > \rho \longrightarrow f'(k) - \delta > \rho \longrightarrow f'(k) > \rho + \delta$$

If the marginal product of capital exceeds its steady state value $(\rho + \delta)$, consumption increases over time. A glance at the figure above reveals that consumption will increase over time for $k < k_{ss}$ and by analogy will decrease over time for $k > k_{ss}$:



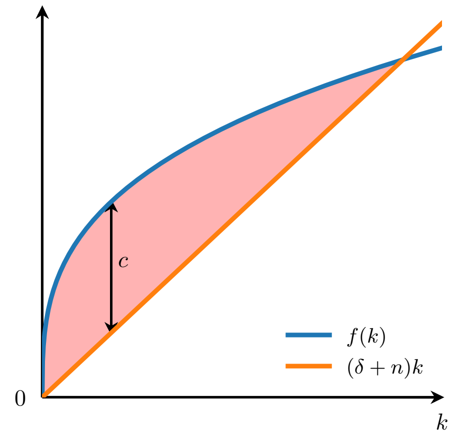
If we rearrange the resource constraint, we will find the expression for c_{ss} given that the capital stock does not change over time:

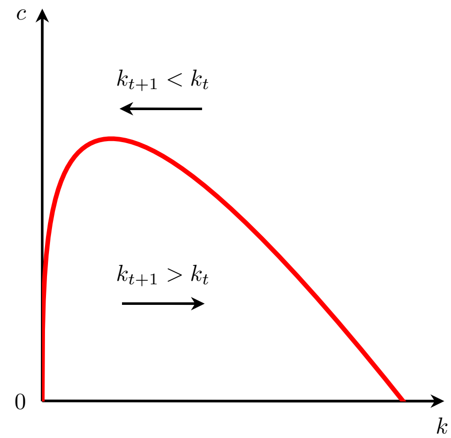
$$\begin{aligned}(1+n)\,k_{ss} &= f(k_{ss}) + (1-\delta)\,k - c_{ss}\\ c_{ss} &= f(k_{ss}) - (\delta + n)\,k_{ss}\end{aligned}$$

Let us see what will happen if the consumption will be chosen above the level guaranteeing stable capital over time:

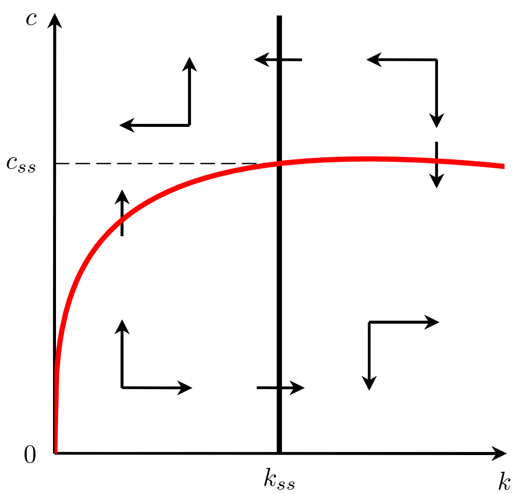
$$\begin{aligned}c_t &> f(k_t) - (\delta + n)\,k_t\\ (1+n)\,k_{t+1} &= f(k_t) + (1-\delta)\,k_t - c_t\\ (1+n)\,k_{t+1} &< f(k_t) + (1-\delta)\,k_t - [f(k_t) - (\delta+n)\,k_t]\\ (1+n)\,k_{t+1} &< (1+n)\,k_t\\ k_{t+1} &< k_t\end{aligned}$$

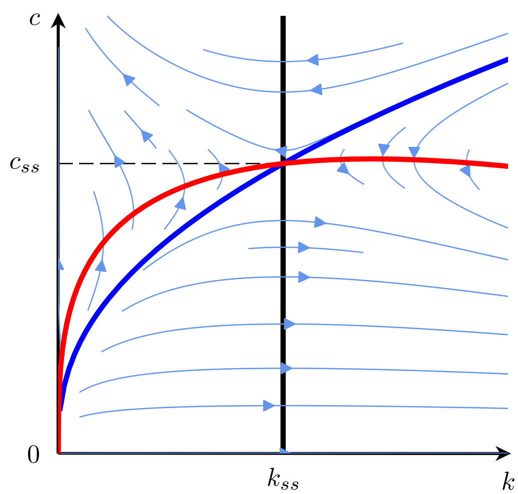
If consumption will be chosen above the quantity that implies constant capital, capital will decrease over time as investment will be lower than depreciation. If consumption will be chosen below the constant capital schedule, capital will increase over time. The relationship is displayed in the graphs below:





We can now join the two schedules and produce the full phase diagram:





1.6 Adding technological progress

Uzawa's Theorem for Balanced Growth Path
If K_t, Y_t and C_t grow at constant rates g_K, g_Y and g_C for all $t \geq T$, population grows at constant rate n, and the "true" production function $\tilde{F}\left(K_t, L_t, \tilde{A}_t\right)$ exhibits constant returns to scale in K_t and L_t, then:

1. $g_K = g_Y = g_C$
2. Production function has a labor-augmenting representation, that is, there exists a technology term A_t that grows at rate $g = g_Y - n$ and a production function F such that

$$\tilde{F}\left(K_t, L_t, \tilde{A}_t\right) = F\left(K_t, A_t L_t\right) \quad \text{for all } t \geq T$$

Sketch of proof[1]
To prove the first proposition, start with the capital accumulation equation and rewrite it to obtain growth rate of capital:

$$\begin{aligned} K_{t+1} &= Y_t - C_t + (1-\delta) K_t \\ \Delta K_{t+1} \equiv K_{t+1} - K_t &= Y_t - C_t - \delta K_t \\ g_{K,t+1} \equiv \frac{\Delta K_{t+1}}{K_t} &= \frac{Y_t}{K_t} - \frac{C_t}{K_t} - \delta \end{aligned}$$

If the growth rate of capital g_K is to be constant, then both the Y/K and C/Y ratios have to be constant over time. If that is the case, then the growth rates of those variables have to be equal.

To prove the second proposition, start with the "true" production function:

$$Y_t = \tilde{F}\left(K_t, L_t, \tilde{A}_t\right)$$

Without loss of generality consider now the more relevant case when population grows at rate $n < g_Y = g_K$ (GDP per capita is increasing over time). Capital grows at the same rate as total output, but pure labor inputs "fall short". The constant returns to scale assumption requires that if one input grows at the same rate as output, the other necessarily has to grow at the same common rate. Thus effective labor input has to be supplemented by some additional factor. Call it technology A. Then it is true that:

$$g_Y = g_K = g_N + g_A \longrightarrow g \equiv g_A = g_Y - n$$

and there exists a representation of the "true" production function $\tilde{F}$ such that:

$$\tilde{F}\left(K_t, L_t, \tilde{A}_t\right) = F\left(K_t, A_t L_t\right) \quad \text{for all } t \geq T$$

Note that the Uzawa's theorem does not mean that all technological improvements increase productivity of labor directly. It only means that technical innovations predominantly are such that not only do labor and capital in combination become more productive, but this manifests itself such that we can rewrite the production function in the form of $F(K, AL)$.

[1] Full proof can be found in e.g. Acemoglu (2009) *Introduction to Modern Economic Growth*, pp. 59-64. You can also see **here**.

From now on, we will assume that technology A grows at constant rate g:

$$A_{t+1} = (1+g)\,A_t$$

Since technology grows over time, both consumption and capital per worker will grow over time. To find the balanced growth path of the system, we need to utilize variables per effective units of labor:

$$\begin{aligned}
\text{Capital per effective labor} &: \quad \hat{k} \equiv \frac{K}{AL} = \frac{k}{A} \\
\text{Consumption per effective labor} &: \quad \hat{c} \equiv \frac{C}{AL} = \frac{c}{A} \\
\text{Output per effective labor} &: \quad \hat{y} \equiv \frac{Y}{AL} = \frac{y}{A} \\
\text{Effective wage} &: \quad \hat{w} \equiv \frac{w}{A}
\end{aligned}$$

The problem of households is unchanged, but now we want to rewrite the Euler equation in terms of per effective labor variables:

$$\begin{aligned}
\left(\frac{c_{t+1}}{c_t}\right)^{\theta} &= \frac{1+r_{t+1}}{1+\rho} \\
\left(\frac{\hat{c}_{t+1}A_{t+1}}{\hat{c}_t A_t}\right)^{\theta} &= \frac{1+r_{t+1}}{1+\rho} \\
\left(\frac{\hat{c}_{t+1}}{\hat{c}_t}\right)^{\theta}(1+g)^{\theta} &= \frac{1+r_{t+1}}{1+\rho} \\
\left(\frac{\hat{c}_{t+1}}{\hat{c}_t}\right)^{\theta} &= \frac{1+r_{t+1}}{1+\rho}\frac{1}{(1+g)^{\theta}} \\
\frac{\hat{c}_{t+1}}{\hat{c}_t} &= \left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta}\frac{1}{1+g}
\end{aligned}$$

Now we have to restate the problem of the firm:

$$\begin{aligned}
\max \quad \Pi_t &= F\left(K_t, A_t L_t\right) - w_t L_t - r_t^k K_t \\
\max \quad \Pi_t &= A_t L_t \left[F\left(\frac{K_t}{A_t L_t}, 1\right) - \frac{w_t}{A_t} - r_t^k \frac{K_t}{A_t L_t}\right] \\
\max \quad \Pi_t &= A_t L_t \left[f\left(\hat{k}_t\right) - \hat{w}_t - r_t^k \hat{k}_t\right]
\end{aligned}$$

where $\hat{y} = f\left(\hat{k}_t\right) \equiv F\left(\frac{K_t}{A_t L_t}, 1\right)$. First order conditions:

$$\begin{aligned}
\hat{k}_t &: \quad A_t L_t \left[f'\left(\hat{k}_t\right) - r_t^k\right] = 0 \\
L_t &: \quad A_t \left[f\left(\hat{k}_t\right) - \hat{w}_t - r_t^k \hat{k}_t\right] = 0
\end{aligned}$$

Simplify and rewrite:

$$\begin{aligned}
r_t^k &= f'\left(\hat{k}_t\right) \\
\hat{w}_t &= f\left(\hat{k}_t\right) - r_t^k \hat{k}_t = f\left(\hat{k}_t\right) - f'\left(\hat{k}_t\right)\hat{k}_t
\end{aligned}$$

Real interest rate:

$$r_t = r_t^k - \delta = f'\left(\hat{k}_t\right) - \delta$$

To find general equilibrium of the economy, start with the households' budget constraint in per capita terms, rewrite in per effective labor terms and plug in prices:

$$\begin{aligned}
(1+n)\, a_{t+1} &= w_t + (1+r_t)\, a_t - c_t \quad | \quad k_t = a_t \\
(1+n)\, k_{t+1} &= w_t + (1+r_t)\, k_t - c_t \quad | \quad : A_t \\
(1+n)\, \frac{k_{t+1}}{A_{t+1}} \frac{A_{t+1}}{A_t} &= \hat{w}_t + (1+r_t)\, \hat{k}_t - \hat{c}_t \\
(1+n)\,(1+g)\, \hat{k}_{t+1} &= f\left(\hat{k}_t\right) - f'\left(\hat{k}_t\right)\hat{k}_t + \left(1 + f'\left(\hat{k}_t\right) - \delta\right)\hat{k}_t - \hat{c}_t \\
(1+n+g+ng)\, \hat{k}_{t+1} &= f\left(\hat{k}_t\right) + (1-\delta)\, \hat{k}_t - \hat{c}_t
\end{aligned}$$

We often approximate and set $ng \approx 0$. The Euler equation after plugging in the interest rate becomes:

$$\left(\frac{\hat{c}_{t+1}}{\hat{c}_t}\right)^\theta = \frac{1 + f'\left(\hat{k}_{t+1}\right) - \delta}{1+\rho} \frac{1}{(1+g)^\theta}$$

The procedure for finding the steady state (balanced growth path) is the same as before. From Euler equation we can obtain the balanced growth path level of capital per effective labor:

$$\begin{aligned}
f'\left(\hat{k}_{ss}\right) &= (1+\rho)\,(1+g)^\theta - (1-\delta) \\
f'\left(\hat{k}_{ss}\right) &\approx (1+\rho)\,(1+\theta g) - 1 + \delta \\
f'\left(\hat{k}_{ss}\right) &\approx \rho + \theta g + \delta
\end{aligned}$$

where the first-order approximation $(1+x)^n \approx 1 + nx$ was used.

Balanced growth path level of consumption per effective labor can be obtained from the resource constraint:

$$(1+n+g+ng)\, \hat{k}_{ss} = f\left(\hat{k}_{ss}\right) + (1-\delta)\, \hat{k}_{ss} - \hat{c}_{ss}$$

$$\hat{c}_{ss} \approx f\left(\hat{k}_{ss}\right) - (\delta + n + g)\, \hat{k}_{ss}$$

To relate it to previous results, imagine that $g = 0$ and technology is normalized to 1: $A = 1$. Then $\hat{k} = k$ and the new expressions reduce to the expressions for the zero technology growth case.

Note how higher rate of technology growth lowers both $\hat{k}_{ss}$ and $\hat{c}_{ss}$. However, you need to remember that those variables do not impact the welfare of agents directly, as households care for their consumption per capita c, which increases faster with higher g. Therefore, higher g increases the welfare of agents.

2 Optimal taxation in the long run

For simplicity, we will assume $n = g = 0$ and $N = A = 1$. We will consider two uses for the taxes – lump-sum transfers to households v and useless from the point of view of households government consumption c^G. We assume that in each period the government's budget is balanced.

2.1 Households

Utility maximization problem:

$$\begin{aligned} \max \quad & U_0 = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\theta} - 1}{1-\theta} \\ \text{subject to} \quad & a_{t+1} = (1 + (1 - \tau_t^a) r_t) a_t + (1 - \tau_t^w) w_t - (1 + \tau_t^c) c_t - \tau_t + v_t \qquad \forall t = 0, 1, \ldots, \infty \end{aligned}$$

where τ^a is capital gains tax, τ^w is labor income tax, τ^c is consumption tax and τ is lump-sum tax.

Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ \frac{c_t^{1-\theta} - 1}{1-\theta} + \lambda_t \left[(1 + (1 - \tau_t^a) r_t) a_t + \left(1 - \tau_t^l\right) w_t - (1 + \tau_t^c) c_t - \tau_t^{ls} + v_t - a_{t+1} \right] \right\}$$

FOCs:

$$\begin{aligned} c_t \quad &: \quad \beta^t \left\{ c_t^{-\theta} - \lambda_t (1 + \tau_t^c) \right\} = 0 \longrightarrow \lambda_t = \frac{c_t^{-\theta}}{1 + \tau_t^c} \\ a_{t+1} \quad &: \quad -\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1} \left(1 + \left(1 - \tau_{t+1}^a\right) r_{t+1}\right) = 0 \end{aligned}$$

Euler equation:

$$\begin{aligned} \frac{c_t^{-\theta}}{1 + \tau_t^c} &= \beta \frac{c_{t+1}^{-\theta}}{1 + \tau_{t+1}^c} \left(1 + \left(1 - \tau_{t+1}^a\right) r_{t+1}\right) \\ \left(\frac{c_{t+1}}{c_t}\right)^{\theta} &= \frac{1 + \tau_t^c}{1 + \tau_{t+1}^c} \beta \left(1 + \left(1 - \tau_{t+1}^a\right) r_{t+1}\right) \end{aligned}$$

2.2 Firms

Profit maximizing problem:

$$\begin{aligned} \max \quad & \Pi_t = \left(1 - \tau_t^f\right) \left[F(K_t, L_t) - \delta K_t - w_t L_t\right] - r_t K_t \qquad \forall t = 0, 1, \ldots, \infty \\ \max \quad & \Pi_t = \left(1 - \tau_t^f\right) L_t \left[f(k_t) - \delta k_t - w_t\right] - r_t L_t k_t \qquad \forall t = 0, 1, \ldots, \infty \end{aligned}$$

where τ^f is firm earnings tax.

FOCs:

$$\begin{aligned} k_t \quad &: \quad \left(1 - \tau_t^f\right) L_t \left[f'(k_t) - \delta\right] - r_t L_t = 0 \longrightarrow r_t = \left(1 - \tau_t^f\right) \left[f'(k_t) - \delta\right] \\ L_t \quad &: \quad \left(1 - \tau_t^f\right) \left[f(k_t) - \delta k_t - w_t\right] - r_t k_t = 0 \\ & \longrightarrow w_t = f(k_t) - \delta k_t - \frac{r_t k_t}{\left(1 - \tau_t^f\right)} = f(k_t) - f'(k_t) k_t \end{aligned}$$

The tax on firm's earnings lowers the return on capital and incentivises firms to hold less capital.

After-tax profits are still zero (because of price taking behavior):

$$\left(1-\tau_t^f\right)\left[f\left(k_t\right)-\delta k_t-\left[f\left(k_t\right)-f'\left(k_t\right)k_t\right]\right]-\left(1-\tau_t^f\right)\left[f'\left(k_t\right)-\delta\right]k_t=$$
$$=\left(1-\tau_t^f\right)\left[-\delta k_t+f'\left(k_t\right)k_t\right]-\left(1-\tau_t^f\right)\left[f'\left(k_t\right)-\delta\right]k_t=0$$

Firm earnings tax revenue is equal to:

$$\tau_t^f\left[f\left(k_t\right)-\delta k_t-\left[f\left(k_t\right)-f'\left(k_t\right)k_t\right]\right]=\tau_t^f\left[f'\left(k_t\right)-\delta\right]k_t$$

2.3 Government sector

The government maintains balanced budget. In per capita terms:

$$c_t^G+v_t=\tau_t^f\left[f\left(k_t\right)-\delta k_t-w_t\right]+\tau_t^a r_t a_t+\tau_t^c c_t+\tau_t^w w_t+\tau_t$$
$$c_t^G+v_t=\tau_t^f\left[f'\left(k_t\right)-\delta\right]k_t+\tau_t^a r_t a_t+\tau_t^c c_t+\tau_t^w w_t+\tau_t$$

2.4 General equilibrium

Market clearing for capital:

$$k_t=a_t$$

Rewrite households' budget constraint to get capital accumulation equation:

$$k_{t+1}=\left(1+\left(1-\tau_t^a\right)r_t\right)k_t+\left(1-\tau_t^w\right)w_t-\left(1+\tau_t^c\right)c_t-\tau_t+v_t$$
$$k_{t+1}=\left(1+r_t\right)k_t+w_t-c_t-\left(\tau_t^a r_t k_t+\tau_t^w w_t+\tau_t^c c_t+\tau_t-v_t\right)$$
$$k_{t+1}=\left(1+\left(1-\tau_t^f\right)\left[f'\left(k_t\right)-\delta\right]\right)k_t+f\left(k_t\right)-f'\left(k_t\right)k_t-c_t-\left(c_t^G-\tau_t^f\left[f'\left(k_t\right)-\delta\right]k_t\right)$$
$$k_{t+1}=f\left(k_t\right)+\left(1-\delta\right)k_t-c_t-c_t^G$$

Rewrite the Euler equation:

$$\left(\frac{c_{t+1}}{c_t}\right)^{\theta}=\frac{1+\tau_t^c}{1+\tau_{t+1}^c}\beta\left(1+\left(1-\tau_{t+1}^a\right)\left(1-\tau_{t+1}^f\right)\left[f'\left(k_{t+1}\right)-\delta\right]\right)$$

2.5 Steady state

Assume constant tax rates:

$$1=\beta\left(1+\left(1-\tau^a\right)\left(1-\tau^f\right)\left[f'\left(k_{ss}\right)-\delta\right]\right)$$
$$\frac{1}{\beta}-1=\left(1-\tau^a\right)\left(1-\tau^f\right)\left[f'\left(k_{ss}\right)-\delta\right]$$
$$\frac{\rho}{\left(1-\tau^a\right)\left(1-\tau^f\right)}=f'\left(k_{ss}\right)-\delta$$
$$f'\left(k_{ss}\right)=\frac{\rho}{\left(1-\tau^a\right)\left(1-\tau^f\right)}+\delta$$

$$k_{t+1}=k_t=k_{ss}\quad\longrightarrow\quad c_{ss}=f\left(k_{ss}\right)-\delta k_{ss}-c^G$$

Government consumption lowers private consumption but does not affect steady state capital per worker.

Capital gains and firm earnings taxes lower steady state capital per worker which then translates to lower steady state private consumption.

2.6 Chamley-Judd result – redistribution impossibility theorem

Again assume $n = g = 0$ and $N = A = 1$ for simplicity. Divide population into two groups: workers and capitalists. Workers do not save and consume their wages and any transfers they receive. Capitalists both save and consume. The government wants to redistribute between capitalists and workers. It levies tax on capital gains and distributes the proceeds to workers.

Worker households

$$\begin{aligned} \max \quad & U_0 = \sum_{t=0}^{\infty} \beta^t \frac{\left(c_t^w\right)^{1-\theta} - 1}{1-\theta} \\ \text{subject to} \quad & c_t^w = w_t + v_t \qquad \forall t = 0, 1, \ldots, \infty \end{aligned}$$

Solution:

$$c_t^w = w_t + v_t$$

Capitalist households

$$\begin{aligned} \max \quad & U_0 = \sum_{t=0}^{\infty} \beta^t \frac{\left(c_t^c\right)^{1-\theta} - 1}{1-\theta} \\ \text{subject to} \quad & a_{t+1} = \left(1 + \left(1 - \tau^a\right) r_t\right) a_t - c_t^c \qquad \forall t = 0, 1, \ldots, \infty \end{aligned}$$

Solution:

$$\left(\frac{c_{t+1}^c}{c_t^c}\right)^{\theta} = \beta \left(1 + \left(1 - \tau^a\right) r_{t+1}\right)$$

Firms

$$\max \quad \Pi_t = L_t \left[f\left(k_t\right) - w_t - r_t^k k_t\right]$$

Solution:

$$\begin{aligned} r_t^k &= f'\left(k_t\right) \longrightarrow r_t = f'\left(k_t\right) - \delta \\ w_t &= f\left(k_t\right) - f'\left(k_t\right) k_t \end{aligned}$$

Government sector

$$v_t = \frac{N^c}{N^w} \tau^a r_{t+1} a_{t+1}$$

General equilibrium

Capital market equilibrium:

$$k_t = \frac{N^c}{N^w} a_t \longrightarrow v_t = \tau^a r_{t+1} k_{t+1}$$

Steady state capital per worker:

$$\begin{aligned} 1 &= \beta \left(1 + \left(1 - \tau^a\right) \left[f'\left(k\right) - \delta\right]\right) \\ f'\left(k\right) &= \frac{\rho}{1 - \tau^a} + \delta \end{aligned}$$

Steady state capitalists' consumption:

$$\begin{aligned} a &= \left(1 + \left(1 - \tau^a\right) r\right) a - c^c \\ c^c &= \left(1 - \tau^a\right) r a = \left(1 - \tau^a\right) r \frac{N^w}{N^c} k \\ c^c &= \left(1 - \tau^a\right) \frac{N^w}{N^c} \left[f'\left(k\right) - \delta\right] k \end{aligned}$$

Steady state workers' consumption:

$$c^w = f(k) - f'(k)\,k + \tau^a\left[f'(k) - \delta\right]k$$

To demonstrate the result, it suffices to show that workers' consumption depends positively on the steady state stock of capital per worker:

$$\begin{aligned}
\frac{\partial c^w}{\partial k} &= f'(k) - \left[f''(k)\,k + f'(k)\right] + \tau^a\left[f''(k)\,k + f'(k) - \delta\right] \\
&= -f''(k)\,k + \tau^a\left[f''(k)\,k + f'(k) - \delta\right] \\
&= \underbrace{(\tau^a - 1)}_{<0}\underbrace{f''(k)}_{<0}k + \tau^a\frac{\rho}{1-\tau^a} > 0
\end{aligned}$$

It turns out that it is impossible to increase steady state consumption of workers by taxing capitalists. Taxing capitalists reduces steady state capital stock and lowers wages. Even if all of the revenue from taxation is given to workers in transfer, the loss in wages is greater than the gain from the transfer.

See e.g. **here** for conditions under which the above result might not hold. For example, **Aiyagari (1995)** shows that with incomplete insurance markets and borrowing constraints, the optimal capital gains tax rate is positive, even in the long run.